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ON THE EXISTENCE OF NOT NECESSARILY UNIQUE SOLUTIONS OF THE CLASSICAL HYPERBOLIC BOUNDARY VALUE PROBLEMS FOR NON-LINEAR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS IN TWO INDEPENDENT VARIABLES.

By

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Thesis

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VITA

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NOTATIONS

The following special notations will be used throughout this paper with the meanings as defined below. Other special notations used will be defined at the place where they are introduced.

ϵ	is a member of; i.e. belongs to.
$B: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \end{cases}$	B is the set of all ordered pairs (x,y), (points) for which $0 \leq x \leq \ell$ and $0 \leq y \leq \ell$.
$f \epsilon C(B)$	f is a member of the class of functions continuous on the set B.
$g \epsilon C'(H)$	g is a member of the class of functions continuously differentiable on the set H, (and similarly for higher degrees of differentiability.)
u_x	$\frac{\partial u}{\partial x}$.
$u_{\lambda,x}$	$\frac{\partial u_\lambda}{\partial x}$.
$\dot{x}$	$\frac{dx}{d\tau}$ where τ is a parameter along a path.
$x \epsilon [0,\ell]$	x belongs to the closed interval, $0 \leq x \leq \ell$.
$\Longrightarrow$	implies.
$\Longleftrightarrow$	implies and is implied by; i.e. if and only if.
$\left\{ g_\lambda \right\} (x,y;\ u;\ p,q)$	a sequence of functions g_λ, $(\lambda = 1,2,\cdots)$, of arguments $(x,y;\ u;\ p,q)$.
$\left\{ g_\lambda \right\} \to f$ on B	the sequence $\left\{ g_\lambda \right\}$ converges pointwise on the set B to the function f.

$\left\{g_\lambda\right\} \xrightarrow{\text{unif}} f$ on B	the sequence $\left\{g_\lambda\right\}$ converges uniformly on the set B to the function f.
$D_\pm y$	the right(+) and left (-) hand derivatives of the function y at the point in question.

CHAPTER I

INTRODUCTION

The purpose of this paper is to present a number of existence theorems pertaining to a class of non-linear second order partial differential equations in two independent variables of the general form

$$(1.1) \qquad F(x,y;\ u;\ p,q;\ r,s,t) = 0,$$

where

$$(1.2) \qquad p = u_x, \quad q = u_y, \quad r = u_{xx}, \quad s = u_{xy} \text{ and } t = u_{yy} ,$$

in the usual notation. We restrict our attention to those prescriptions of initial conditions for which integral surfaces exist such that the equation is of hyperbolic type thereon, i.e. the inequality

$$(1.3) \qquad F_s^2 - 4\, F_r\, F_t > 0$$

must be satisfied on the integral surface in a neighborhood of the initial data.

E. PICARD [1],[7][1], E. GOURSAT [8], E.E.Levi[9], H.LEWY[10], J. HADAMARD[11], M. CINQUINI-CIBRARIO[12],[13], and others have

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[1] The number in the bracket [ ] refers to the reference in the bibliography.

developed existence theorems based on the method of successive approximations. Their concern has been to establish sufficient conditions for the existence of a unique solution. Retaining their restrictions on the initial data, we shall obtain sufficient conditionsfor the existence of at least one solution. The integrals of the equations we consider will not, in general, be unique.

The concept of characteristic curves in an integral surface plays an important role in all work in this field. We give two definitions of a characteristic curve, the first applicable when the curve is expressed in non-parametric form, the second when expressed in parametric form:

Definition 1

$\gamma: \begin{cases} a \leq x \leq b \\ y = g(x) \end{cases}$ where $g \in C'([a,b])$, or $\gamma: \begin{cases} x = h(y) \\ c \leq y \leq d \end{cases}$

where $h \in C'([c,d])$, is a characteristic base curve (characteristic projection or, by usage, characteristic) for a particular integral surface J: $u=u(x,y)$ of $F(x,y;\ u;\ p,q;\ r,s,t) = 0 \Longleftrightarrow$ for each (x,y)

$$(1.4) \qquad F_r dy^2 - F_s dydx + F_t dx^2 = 0$$

Definition 1a

$\gamma: \begin{cases} x=x(\tau) \\ y=y(\tau) \end{cases}$ for $\tau \in [0,1]$ and where $x, y \in C'([0,1])$, is a characteristic base curve for a particular integral surface J: $u = u(x,y)$ of $F(x,y;\ u;\ p,q;\ r,s,t) = 0 \Longleftrightarrow$ for each $\tau \in [0,1]$

$$(1.5) \qquad \begin{cases} 1) \quad F_r \dot{y}^2 - F_s \dot{y}\dot{x} + F_t \dot{x}^2 = 0 \\ 2) \quad \dot{x}^2 + \dot{y}^2 \neq 0. \end{cases}$$

Under either definition Υ is rectifiable and possesses a continuously turning tangent (see C. JORDAN[6], p. 100). The two definitions are equivalent in the following sense: We may convert Υ expressed in non-parametric form into its parametric expression by setting $x = \tau$, $y = g(\tau)$, or $x = h(\tau)$, $y = \tau$ as the case may be. That the converse is possible follows directly from condition 2) of Definition 1a and the Implicit Function Theorem. For, suppose at a point $(x(\tau_0), y(\tau_0))$ of Υ that $\dot{x} \neq 0$. Then in a vicinity of $x_0 = x(\tau_0)$ the inverse relation $\tau = \tau(x)$ exists and we may write

$$(1.6) \qquad \Upsilon \;:\; y = y(\tau(x)) = g(x).$$

Similarly, where $\dot{y} \neq 0$, we may write

$$(1.7) \qquad \Upsilon \;:\; x = x(\tau(y)) = h(y).$$

By condition 2), one of the two representations (1.6) or (1.7) is always possible in the vicinity of each point of Υ.

Definition 2

$$\Gamma \;:\; \begin{cases} x = x(\tau) \\ y = y(\tau) \\ u = u(\tau) \end{cases} \text{ for } \tau \in [0,1] \text{ and where } x,y,u \in C'([0,1]),$$

a space curve lying in a particular integral surface J: $u=u(x,y)$ of $F(x,y;\, u;\, p,q;\, r,s,t) = 0$, is called a characteristic curve in the integral surface J $\Longleftrightarrow$ the projection of Γ onto the xy plane is a characteristic projection for the integral surface J.

Under suitable hypotheses, by virtue of the hyperbolic condition (1.3), for any integral surface J: $u=u(x,y)$ of $F(x,y;u;p,q;,r,s,t)=0$, equations (1.4) or (1.5) determine two one parameter families of characteristic curves lying in the integral surface J. Exactly one characteristic curve from each family passes through any given point $(x_0,y_0,u(x_0,y_0))$ of the integral surface J; and, moreover, the corresponding two characteristic base curves do not have a common tangent at (x_0,y_0).

Along any curve, characteristic or otherwise, lying in the integral surface J, the following strip, or band, conditions

$$(1.8) \qquad \dot{u} = p\dot{x} + q\dot{y}$$

$$(1.9) \qquad \begin{cases} \dot{p} = r\dot{x} + s\dot{y} \\ \dot{q} = s\dot{x} + t\dot{y} \end{cases}$$

must be satisfied.

The modification of Definition 2 and conditions (1.8),(1.9) when the curve Γ is expressed in non-parametric form is obvious.

Definition 3

$$S^1: \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \\ p=p(\tau) \\ q=q(\tau) \end{cases} \quad \text{for } \tau \in [0,1] \text{ and where } x,y,u,p,q \in C'([0,1]).$$

is called a first order strip $\Longleftrightarrow$ for each $\tau \in [0,1]$

$$(1.8) \qquad \dot{u} = p\dot{x} + q\dot{y}$$

Suppose a particular integral surface J: $u=u(x,y)$ of

$F(x,y;\ u;\ p,q;\ r,s,t) = 0$ has a contact of first order with the strip S^1. Then if $\Gamma : \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \end{cases}$ for $\tau \in [0,1]$ is a characteristic curve in the integral surface J, the strip S^1 is called a characteristic first order strip for the integral surface J.

Definition 4

$$S^2 : \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \\ p=p(\tau) \\ q=q(\tau) \\ r=r(\tau) \\ s=s(\tau) \\ t=t(\tau) \end{cases} \quad \text{for } \tau \in [0,1] \text{ and where } x,y,u,p,q,r,s,t \in C'([0,1])$$

is called a second order strip $\Longleftrightarrow$ for each $\tau \in [0,1]$

$$(1.8) \qquad \dot{u} = p\dot{x} + q\dot{y}$$

$$(1.9) \qquad \begin{cases} \dot{p} = r\dot{x} + s\dot{y} \\ \dot{q} = s\dot{x} + t\dot{y} \end{cases}$$

If, moreover, equation (1.1) and conditions (1.3) and (1.5) are satisfied for each $\tau \in [0,1]$, then S^1 is called a characteristic second order strip.

Note in Definition 4 that since all the arguments of the functions involved in conditions (1.5) are known upon prescription of the strip S^2, we may determine whether or not the projection of corresponding space curve $\Gamma : \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \end{cases}$ for $\tau \in [0,1]$ is a characteristic projection without reference to any particular integral surface.

Definitions 3 and 4 can be readily modified to deal with the non-parametric case. See, for example, M. CINQUINI-CIBRARIO[13].

In Chapter 2 we consider the characteristic initial value problem for the equation

$$(1.10) \qquad s = f(x,y;\ u;\ p,q)$$

and its extension to the system of equations

$$(1.11) \qquad s_i = f_i(x,y;\ u_1,\cdots,u_n;\ p_1,\cdots,p_n,\ q_1,\cdots,q_n) \quad (i=1,2,\cdots,n).$$

We modify the customary hypothesis that f be Lipschitzian, i.e. with respect to variables u, p and q, to require that f be partially Lipschitzian, i.e. with respect to variables p and q only. We obtain existence of an integral u over the same closed domain as that obtained in the classical theory. Our integral, however, cannot be shown to be unique. This fact is demonstrated by an example. By further example, we show that the bounds obtained on the domain of existence are maximal bounds.

In Chapter 3 we apply the methods of Chapter 2 to the Cauchy problem for equation (1.10) and the extension to the system (1.11). The conclusions are similar to those obtained in Chapter 2.

The arguments in Chapter 4 serve to establish the equivalence (as defined therein) between the characteristic initial value and the Cauchy problems for the system (1.11) and the corresponding problems for a particular system of first order partial differential equations of the form

$$(1.12)\quad \begin{cases} \sum_{k=1}^{n} A_{ik}u_{k,x} = C_i & (i = 1,2,\cdots,m< n) \\ \sum_{k=1}^{n} A_{ik}u_{k,y} = C_i & (i = m+1,m+2,\cdots,n) \end{cases}$$

where the A_{ik}, C_i are functions of $x,y,u_1,u_2,\cdots,u_n$. The system (1.12) is termed a canonical hyperbolic system.

This equivalence has already been established by M. CINQUINI-CIBRARIO[12]. Under the restriction that the first partial derivatives of the functions A_{ik}, C_i be Lipschitzian with respect to all their arguments, she obtains her theorems on the existence and uniqueness of the system of functions U_i as the solution for the canonical hyperbolic system (1.12). We demonstrate that her reasoning establishing the equivalence does not depend upon the uniqueness of the solutions for either system (1.11) or system (1.12). Consequently, from our results in Chapters 2 and 3, we are able to remove the above Lipschitz condition entirely and obtain existence, but not uniqueness, for the solutions of the canonical hyperbolic system for both characteristic and Cauchy initial value prescriptions.

Following the attack of H. LEWY[10], in Chapter 5 we reduce the equation

$$(1.1)\qquad F(x,y;\ u;\ p,q;\ r,s,t) = 0$$

to a system of so-called characteristic equations by means of a transformation to the characteristic base curves as coordinates. This system is shown to contain a canonical hyperbolic system.

We treat the Cauchy problem, i.e. to find an integral surface which has a second order contact with a prescribed second order strip. By virtue of a theorem by M. CINQUINI-CIBRARIO, stated in Chapter 4, LEWY'S work yields immediately the result that for $F \in C'''$ in a suitable region, there exists a unique solution $u \in C'''$ in a sufficiently small neighborhood of the initial curve. We again demonstrate that the equivalence of the problems is not dependent upon uniqueness of their respective solutions; and, hence, by requiring simply that $F \in C''$ we obtain existence but not uniqueness.

In Chapter 6 we treat the characteristic initial value problem for equation (1.1). We follow a modification of H. LEWY'S method introduced by M. CINQUINI-CIBRARIO[13]. Here equation (1.1) is first transformed into the form

$$s = f(x,y;\ u;\ p,q;\ r,t). \tag{1.13}$$

A modified system of characteristic equations is obtained. This system also contains a canonical hyperbolic system. The theorems of Chapter 2 apply and we obtain results similar to those obtained in Chapter 5 for the Cauchy problem.

In Chapter 7 we treat the mixed boundary value problem for the equation

$$s = f(x,y;\ u;\ p,q), \tag{1.10}$$

i.e. the problem where any integral surface of (1.10) is required to pass through two space curves issuing from a point, with one of the curves being a characteristic on this surface and the other

curve having curve having nowhere a characteristic projection. We show that for equation (1.10) there is no loss in generality if we assume the initial data to be

$$u(x,0) = u(x,x) = 0. \tag{1.14}$$

For f continuous, bounded and Lipschitzian, we prove that there exists one and only one integral surface of (1.10) satisfying (1.14) on a domain for which we prescribe explicit bounds. For f continuous, bounded and partially Lipschitzian, we find, by arguments analogous to those used in Chapters 2 and 3, that there exists at least one integral surface of (1.10) satisfying (1.14) on a domain for which we again prescribe the same type of explicit bounds.

In Chapter 8 we consider the characteristic initial value problem for equation (1.10) from a new point of view. Here, in order to extend the theorems of Chapter 2, we introduce the concept of upper and lower bounding functions for the solution (or solutions) of the problem. This idea was first used by O. PERRON [18] to obtain an existence proof for the problem

$$y' = f(x,y) \quad , \quad y(x_0) = y_0. \tag{1.15}$$

His proof is quite independent of the classical proofs.

M. MÜLLER [4] shows that PERRON's method has no direct analogue for a system

$$y_i' = f_i(x,y_1,\cdots y_n) \quad , \quad (i = 1,\cdots,n). \tag{1.16}$$

He is able, however, to extend the classical theorem for a system (1.16) to obtain a theorem which reduces to the direct analogue to the PERRON theorem in the case where the f_i are monotonically increasing functions of the arguments $y_1,\cdots,y_n$.

The extensions to the theorems of Chapter 2 which we obtain are similar to MÜLLER's conclusions for the system (1.16). Moreover, we demonstrate by example that the PERRON method has no direct analogue for the characteristic initial value problem for equation (1.10). We also give an example illustrating the theorems obtained in this chapter. Finally, we note that the Cauchy problem for equation (1.10) and the Cauchy and characteristic initial value problems for the system

$$(1.11) \qquad s_i = f_i(x,y;\ u_1,\cdots,u_n;\ p_1,\cdots,p_n,\ q_1,\cdots q_n) \qquad (i = 1,\cdots,n),$$

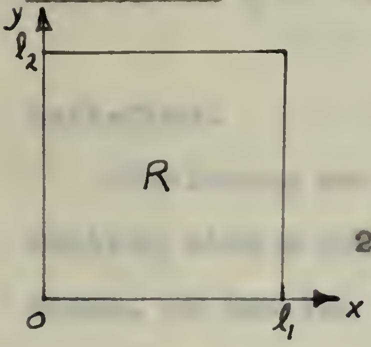
may also be treated by the methods of this chapter.

CHAPTER II

The Characteristic Initial Value Problem for $u_{xy} = f(x,y;u;u_x,u_y)$.

For convenience of reference we first state the following theorem, whose proof is based on the method of successive approximations. The proof of existence was given by É. PICARD [1], while the proof of uniqueness may be found in E. KAMKE [2] p. 410.

Theorem 1. 1) $f(x,y;u;p,q) \in C(B), B: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \\ -a \leq u \leq a \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{cases}$



2) f is Lipschitzian on B; i.e. there exists a positive constant K such that for $(x,y;u_1;p_1,q_1) \in B$, $(x,y;u_2;p_2,q_2) \in B$,

$$\left| f(x,y;u_1;p_1,q_1) - f(x,y;u_2;p_2,q_2) \right| \leq K \left\{ |u_1-u_2| + |p_1-p_2| + |q_1-q_2| \right\}$$

3) $M \ell_1 \ell_2 \leq a$, $M \ell_1 \leq b_2$, $M \ell_2 \leq b_1$, where $M = \max |f|$ on B.

$\Longrightarrow$ 4) There exists one and only one function $u(x,y) \in C'(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$, such that for each $(x,y) \in R$ the point $(x,y;\, u(x,y);\, u_x(x,y),\, u_y(x,y)) \in B$, and $u_{xy}(x,y) = f(x,y;\, u(x,y);\, u_x(x,y),\, u_y(x,y))$, $u(x,0) = 0$, $u(0,y) = 0$ for each $(x,y) \in R$.

Remarks. a) Suppose we prescribe $u(x,0) = U(x)$, $u(0,y) = V(y)$ where $U(x) \in C'([0,\ell_1])$, $V(y) \in C'([0,\ell_2])$ and $U(0) = V(0)$. Consider the function $w(x,y) = U(x) + V(y) - U(0)$. Clearly, $w_{xy}(x,y) = 0$ and $w(x,0) = U(x)$, $w(o,y) = V(y)$ hence the function $v = u - w$ must satisfy $v_{xy} = f(x,y;\ v + w;\ v_x + w_x,\ v_y + w_y)$, $v(x,0) = v(0,y) = 0$, a problem of the type covered by Theorem 1.

b) Suppose $f \in C$, bounded and Lipschitzian in the domain B' : $\begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \\ -\infty < u < \infty \\ -\infty < p < \infty \\ -\infty < q < \infty \end{cases}$ Then hypothesis 3) is immediately satisfied.

Following an approach used by M. MULLER [3] p. 632, in dealing with a system of first order ordinary differential equations, we are led to this improvement of the above theorem:

Theorem 1a. 1)

2)' f is partially Lipschitzian on B; i.e. there exists a positive constant K such that for $(x,y;\ u;\ p_1,q_1) \in B$, $(x,y;\ u;\ p_2,q_2) \in B$, $|f(x,y;\ u;\ p_1,q_1) - f(x,y;\ u;\ p_2,q_2)| \leq K\left\{|p_1-p_2| + |q_1-q_2|\right\}$.

3)

$\Rightarrow$ 4)' There exists at least one function $u(x,y) \in C'(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$ such that for each $(x,y) \in R$

the point $(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)) \in B$, and $u_{xy}(x,y) = f(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y))$, $u(x,0) = 0$, $u(0,y) = 0$ for each $(x,y) \in R$.

Proof. According to WEIERSTRASS' celebrated theorem [4] p. 1147, on polynomial approximations to a continuous function, there exists a sequence of polynomials, $\{g_\lambda\}(x,y;u;p,q)$, converging uniformly to $f(x,y;\ u;\ p,q)$ on B. We designate this uniform convergence by the notation $\{g_\lambda\} \xrightarrow{\text{unif}} f$ on B.

We extend f and the polynomials g_λ, $(\lambda = 1,2,\cdots)$, over the domain B to the domain B', defined in the remark b) above, by the definition

$$f(x,y;\ u;\ p,q) = f(x,y;\ \overline{u};\ \overline{p},\overline{q})$$

$$g\ (x,y;\ u;\ p,q) = g\ \ (x,y;\ \overline{u};\ \overline{p},\overline{q}), \quad (\ \ = 1,2,\cdots),$$

(2.1) where

$\overline{u} = u$ if $-a \leq u \leq a$, $\overline{p} = p$ if $-b_1 \leq p \leq b_1$, $\overline{q} = q$ if $-b_2 \leq q \leq b_2$.

$\overline{u} = a$ if $a < u$ $\quad\overline{p} = b_1$ if $b_1 < p$ $\quad\overline{q} = b_2$ if $b_2 < q$

$\overline{u} = -a$ if $u < -a$ $\quad\overline{p} = -b_1$ if $p < -b_1$ $\quad\overline{q} = -b_2$ if $q < -b_2$

From this extended definition we see that $|f| \leq M$ in B'. Since $\{g_\lambda\} \xrightarrow{\text{unif}} f$ in B', there exists a constant $L > 0$ such that $|g_\lambda| \leq L$ in B' and for all λ. The functions g_λ, $(\lambda = 1,2,\cdots)$ are uniformly continuous in B', moreover they possess bounded difference quotients with respect to the arguments u, p and q everywhere in B'. Hence in B', for each function g_λ there exists a constant $K_\lambda > 0$ such that

$$(2.2)\quad |g_\lambda(x,y;\, u_1;\, p_1,q_1) - g_\lambda(x,y;\, u_2;\, p_2,q_2)| \leq K_\lambda\{|u_1-u_2| + |p_1-p_2| + |q_1-q_2|\}.$$

Thus, by Theorem 1, to each g_λ there corresponds one and only one function $u_\lambda(x,y) \in C'(R)$, $u_{\lambda,xy}(x,y) \in C(R)$ satisfying

$$(2.3)\quad \begin{cases} u_{\lambda,xy} = g_\lambda(x,y;\, u_\lambda(x,y);\, u_{\lambda,x}(x,y),\, u_{\lambda,y}(x,y)), \\ u_\lambda(x,0) = 0, \quad u_\lambda(0,y) = 0 \quad \text{for each } (x,y) \in R. \end{cases}$$

We may express the characteristic initial value problem for each u_λ in the form of an equivalent integral equation

$$(2.4)\quad u_\lambda(x,y) = \int_0^x d\xi \int_0^y g_\lambda(\xi,\eta;\, u_\lambda(\xi,\eta);\, u_{\lambda,x}(\xi,\eta),\, u_{\lambda,y}(\xi,\eta))\,d\eta.$$

By differentiation,

$$(2.5)\quad u_{\lambda,x}(x,y) = \int_0^y g_\lambda(x,\eta;\, u_\lambda(x,\eta);\, u_{\lambda,x}(x,\eta),\, u_{\lambda,y}(x,\eta))\,d\eta$$

$$(2.6)\quad u_{\lambda,y}(x,y) = \int_0^x g_\lambda(\xi,y;\, u_\lambda(\xi,y);\, u_{\lambda,x}(\xi,y),\, u_{\lambda,y}(\xi,y))\,d\xi.$$

We now show that the sequences $\{u_\lambda\}$, $\{u_{\lambda,x}\}$, $\{u_{\lambda,y}\}$ are each uniformly bounded and equicontinuous on R. For the sequence $\{u_\lambda\}$ this follows directly from the integral expression (2.4), for, given x, x_1, $x_2 \in [0,\ell_1]$ and y, y_1, $y_2 \in [0,\ell_2]$,

$$(2.7)\quad |u_\lambda(x,y)| \leq L\,\ell_1\ell_2, \qquad (\lambda = 1,2,\cdots)$$

$$(2.8)\quad |u_\lambda(x_1,y_1) - u_\lambda(x_2,y_2)| \leq L|x_1-x_2|\;|y_1-y_2| + L\,\ell_2\,|x_1-x_2| + L\,\ell_1\,|y_1-y_2|, \quad (\lambda = 1,2,\cdots)$$

The uniform boundedness of $\{u_{\lambda,x}\}$ and of $\{u_{\lambda,y}\}$ follow directly from (2.5) and (2.6), respectively, for, given $(x,y) \in R$,

$$(2.9) \qquad |u_{\lambda,x}(x,y)| \le L\,\ell_2, \qquad (\lambda = 1,2,\cdots)$$

$$(2.10) \qquad |u_{\lambda,y}(x,y)| \le L\,\ell_1, \qquad (\lambda = 1,2,\cdots).$$

We base the proof of the equicontinuity of the functions of the sequence $\{u_{\lambda,x}\}$ upon the following two lemmas, the first of which is due to T. H. GRONWALL [5].

Lemma 1. 1) $Z(y) \in C([0,\ell])$

(2.11) 2) $0 \le Z(y) \le \int_0^y (MZ(\eta) + A)\,d\eta + B$ for $y \in [0,\ell]$
where M, A and B are constants ≥ 0.

(2.12) 3) $0 \le Z(y) \le (A\ell + B)\,e^{M\ell}$ for $y \in [0,\ell]$.

Lemma 2. Given $\mu > 0$, $\zeta > 0$, there exist δ, a positive constant depending upon μ alone, and N, a positive integer depending upon ζ alone, such that whenever $(x_1,y) \in R$, $(x_2,y) \in R$, $|x_1 - x_2| < \delta$ and $\lambda > N$,

$$(2.13) \qquad |u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y)| \le K\int_0^y |u_{\lambda,x}(x_2,\eta) - u_{\lambda,x}(x_1,\eta)|\,d\eta + \mu + \zeta$$

where K is the partial Lipschitz constant for $f(x,y;\,u;\,p,q)$.

Assume, for the moment, the validity of these two lemmas. Each of the functions $u_{\lambda,x}$ is certainly uniformly continuous on R; hence, if we let $Z(y) = |u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y)|$ for any particular $\lambda > N$,

we have immediately that for $|x_2 - x_1| < \delta$,

(2.14) $|u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y)| \leq (\mu + \zeta)\, e^{K\gamma_2}$.

Suppose $(x_1, y_1) \in R$, $(x_2, y_2) \in R$, then certainly $(x_2, y_1) \in R$ and

(2.15) $|u_{\lambda,x}(x_2,y_2) - u_{\lambda,x}(x_1,y_1)| \leq |u_{\lambda,x}(x_2,y_2) - u_{\lambda,x}(x_2,y_1)|$

$+ |u_{\lambda,x}(x_2,y_1) - u_{\lambda,x}(x_1,y_1)|$, $(\lambda = 1,2,\cdots)$.

By (2.5) we have that

(2.16) $|u_{\lambda,x}(x_2,y_2) - u_{\lambda,x}(x_2,y_1)| \leq L\,|y_2 - y_1|$, $(\lambda = 1,2,\cdots)$.

Inequalities (2.14), (2.15) and (2.16) yield immediately the equicontinuity on R of the functions of the sequence $\{u_{\lambda,x}\}$; for, given $\epsilon > 0$, we first choose $\mu > 0$ and $\zeta > 0$ such that

(2.17) $$\mu + \zeta < \frac{\epsilon}{2e^{K\gamma_2}}$$

and let δ and N be the corresponding constants given by Lemma 2. By the uniform continuity on R of each of the functions $u_{\lambda,x}$, there exists a positive constant δ_N, depending on ϵ alone, such that

$$|x_1 - x_2| < \delta_N \text{ and } |y_1 - y_2| < \delta_N \Rightarrow$$

(2.18) $|u_{\lambda,x}(x_2,y_2) - u_{\lambda,x}(x_1,y_1)| < \epsilon$, $(\lambda = 1,2,\cdots, N)$.

Setting $\delta_0 = \min(\delta, \delta_N, \frac{\epsilon}{2L})$ we obtain

$$|x_1-x_2|<\delta_0 \quad \text{and} \quad |y_1-y_2|<\delta_0 \Rightarrow$$

$$|u_{\lambda,x}(x_2,y_2)-u_{\lambda,x}(x_1,y_1)|<\epsilon, \text{ for } \lambda = 1,2,\cdots,N,N+1,\cdots \tag{2.19}$$

Proof of Lemma 1: Let $Z(y) = e^{My}\cdot w(y)$, without loss for we may always choose $w(y) = e^{-My}\cdot Z(y)$. $w(y) \in C([0,\ell])$ and hence attains a maximum thereon. Let w_{max} occur at $y = y_1$, then

$$\begin{aligned} 0 \le e^{My_1} w_{max} &\le \int_0^{y_1}(M e^{M\eta} w(\eta) + A)\, d\eta + B \\ &\le w_{max}\int_0^{y_1} M e^{M\eta}\, d\eta + A y_1 + B \\ &= w_{max}(e^{My_1}-1) + A y_1 + B \end{aligned}$$

Thus $0 \le w_{max} \le A y_1 + B \le A\ell + B$ and hence

$$0 \le Z(y) \le (A\ell + B)\, e^{M\ell} \quad \text{for } y \in [0,\ell].$$

Proof of Lemma 2:

$$\begin{aligned} u_{\lambda,x}(x_2,y)-u_{\lambda,x}(x_1,y) &= \int_0^y [g_\lambda(x_2,\eta;\, u_\lambda(x_2,\eta);\, u_{\lambda,x}(x_2,\eta),\, u_{\lambda,y}(x_2,\eta)) \\ &\quad - g_\lambda(x_1,\eta;\, u_\lambda(x_1,\eta);\, u_{\lambda,x}(x_1,\eta),\, u_{\lambda,y}(x_1,\eta))]\, d\eta \\ &= \int_0^y [g_\lambda(x_2,\eta;\, u_\lambda(x_2,\eta);\, u_{\lambda,x}(x_2,\eta),\, u_{\lambda,y}(x_2,\eta)) \\ &\quad - f(x_2,\eta;\, u_\lambda(x_2,\eta);\, u_{\lambda,x}(x_2,\eta),\, u_{\lambda,y}(x_2,\eta))]\, d\eta \\ &\quad + \int_0^y [f(x_2,\eta;\, u_\lambda(x_2,\eta);\, u_{\lambda,x}(x_2,\eta),\, u_{\lambda,y}(x_2,\eta)) \end{aligned} \tag{2.20}$$

(2.20)
(Continued)

$$- f(x_2,\eta;\ u_\lambda(x_2,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_2,\eta)\Big]\, d\eta$$

$$+\int_0^y \Big[f(x_2,\eta;\ u_\lambda(x_2,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_2,\eta))$$

$$- f(x_1,\eta;\ u_\lambda(x_1,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_1,\eta))\Big]\ d\eta$$

$$+\int_0^y \Big[f(x_1,\eta;\ u_\lambda(x_1,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_1,\eta))$$

$$- g_\lambda(x_1,\eta;\ u_\lambda(x_1,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_1,\eta))\Big]\ d\eta$$

$$(\lambda = 1,2,\cdots\ ).$$

Since $\{g_\lambda\} \xrightarrow{\text{unif}} f$ on B', given $\zeta > 0$, there exists a positive integer N, depending upon ζ alone, such that for $\lambda > N$,

$$(2.21)\ \Big|\int_0^y \Big[g_\lambda(x_2,\eta;\ u_\lambda(x_2,\eta);\ u_{\lambda,x}(x_2,\eta),\ u_{\lambda,y}(x_2,\eta)) - f(x_2,\eta;\ u_\lambda(x_2,\eta);\ u_{\lambda,x}(x_2,\eta),\ u_{\lambda,y}(x_2,\eta))\Big]\ d\eta\Big|$$
$$+\Big|\int_0^y \Big[f(x_1,\eta;\ u_\lambda(x_1,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_1,\eta)) - g_\lambda(x_1,\eta;\ u_\lambda(x_1,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_1,\eta))\Big]\, d\eta\Big| < \zeta$$

By hypothesis 2)',

$$(2.22)\ \Big|\int_0^y \Big[f(x_2,\eta;\ u_\lambda(x_2,\eta);\ u_{\lambda,x}(x_2,\eta),\ u_{\lambda,y}(x_2,\eta)) -$$

(2.22)
(Continued) $-f(x_2,\eta;\ u_\lambda(x_2,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_2,\eta))]\ d\eta|$

$$\leq K\int_0^y |u_{\lambda,x}(x_2,\eta) - u_{\lambda,x}(x_1,\eta)|\ d\eta\ , \quad (\lambda = 1,2,\cdots)$$

Since f is uniformly continuous on R', the functions of the sequence $\{u_\lambda\}$ are equicontinuous on R, and $|u_{\lambda,y}(x_2,\eta) - u_{\lambda,y}(x_1,\eta)| \leq L\,|x_2-x_1|$, $(\lambda = 1,2,\cdots)$, it follows that given $\mu > 0$ there exists a positive constant δ, depending upon μ alone, such that for $|x_2 - x_1| < \delta$,

$$(2.23)\quad \left|\int_0^y \left[f(x_2,\eta;\ u_\lambda(x_2,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_2,\eta)) - f(x_1,\eta;\ u_\lambda(x_1,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_1,\eta))\right] d\eta\right| < \mu,$$

$(\lambda = 1,2,\cdots)$.

Therefore, from (2.21), (2.22) and (2.23), by virtue of (2.20) we obtain that for $\lambda > N$ and $|x_2 - x_1| < \delta$,

$$(2.13)\quad |u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y)| < K\int_0^y |u_{\lambda,x}(x_2,\eta) - u_{\lambda,x}(x_1,\eta)|\ d\eta + \mu + \zeta$$

thus verifying Lemma 2.

The proof of the equicontinuity of the functions of the sequence $\{u_{\lambda,y}\}$ follows precisely the same steps as that for the sequence $\{u_{\lambda,x}\}$.

We now invoke the well-known theorem of C. ARZELA [3] p. 1144:

"Given a set F of functions f defined and continuous on the closed bounded set A, then the necessary and sufficient conditions that each sequence of functions contained in F possesses

a subsequence uniformly convergent on A are that F be uniformly bounded and equicontinuous."

By Theorem 1, there exists a unique triple $(u_\lambda ; u_{\lambda,x}; u_{\lambda,y})$ corresponding to g_λ for each λ . Since any subsequence of a uniformly convergent sequence is likewise uniformly convergent; and, since any subsequence of a uniformly bounded and equicontinuous sequence is likewise uniformly bounded and equicontinuous; there exists a subsequence $\left\{ g^*_\lambda \right\}$ of the sequence $\left\{ g_\lambda \right\}$ such that the corresponding sequences

$$(2.24) \quad \left\{ u^*_\lambda \right\} \xrightarrow{\text{unif}} u, \quad \left\{ u^*_{\lambda,x} \right\} \xrightarrow{\text{unif}} v, \quad \left\{ u^*_{\lambda,y} \right\} \xrightarrow{\text{unif}} w,$$

where $u, v, w \in C(R)$. This results from the following successive extractions of subsequences:

$\left\{ u_\lambda \right\}$ is equicontinuous and uniformly bounded on R, hence there exists a subsequence $\left\{ u^1_\lambda \right\}$ of $\left\{ u_\lambda \right\}$ uniformly convergent on R. $\left\{ u^1_{\lambda,x} \right\}$ is equicontinuous and uniformly bounded on R, hence there exists a subsequence $\left\{ u^2_{\lambda,x} \right\}$ of $\left\{ u^1_{\lambda,x} \right\}$ uniformly convergent on R. $\left\{ u^2_{\lambda,y} \right\}$ is equicontinuous and uniformly bounded on R, hence there exists a subsequence $\left\{ u^*_{\lambda,y} \right\}$ of $\left\{ u^2_{\lambda,y} \right\}$ uniformly convergent on R. But, by the one-to-one correspondence mentioned above, $\left\{ u^*_{\lambda,x} \right\}$ is a subsequence of $\left\{ u^2_{\lambda,x} \right\}$ while $\left\{ u^*_\lambda \right\}$ is a subsequence of $\left\{ u^1_\lambda \right\}$. Thus $\left\{ u^*_{\lambda,x} \right\}$ and $\left\{ u^*_\lambda \right\}$ are each uniformly convergent on R.

Writing, with the notation $u^*_0 = u^*_{0,x} = u^*_{0,y} = 0$,

$$(2.25)\quad u^*_\lambda = \sum_{k=1}^{\lambda}(u^*_k - u^*_{k-1}),\ u^*_{\lambda,x} = \sum_{k=1}^{\lambda}(u^*_{k,x} - u^*_{k-1,x}),$$

$$u^*_{\lambda,y} = \sum_{k=1}^{\lambda}(u^*_{k,y} - u^*_{k-1,y}),\quad (\lambda = 1,2,\cdots),$$

we see that the conditions for differentiation under the summation sign for infinite series are satisfied by (2.24) and the fact that $u^*_\lambda \in C'(R)$, $(\lambda = 1,2,\cdots)$. Hence

$$(2.26)\quad v(x,y) = u_x(x,y),\quad w(x,y) = u_y(x,y)\quad \text{for } (x,y) \in R$$

We show that the function u so determined satisfies the integral equation equivalent to the original characteristic initial value problem, i.e.

$$(2.27)\quad u(x,y) = \int_0^x d\xi \int_0^y f(\xi,\eta;\, u(\xi,\eta);\, u_x(\xi,\eta),\, u_y(\xi,\eta))\,d\eta$$

for $(x,y) \in R$.

For any λ, by (2.4),

$$\begin{aligned}(2.28)\quad &\left|u(x,y) - \int_0^x d\xi \int_0^y f(\xi,\eta;\, u(\xi,\eta);\, u_x(\xi,\eta),\, u_y(\xi,\eta))\,d\eta\right| \\ &\leq \left|u(x,y) - u^*_\lambda(x,y)\right| + \int_0^x d\xi \int_0^y \left| f(\xi,\eta;\, u(\xi,\eta);\, u_x(\xi,\eta),\right. \\ &\qquad \left. u_y(\xi,\eta)) - f(\xi,\eta;\, u^*_\lambda(\xi,\eta);\, u^*_{\lambda,x}(\xi,\eta),\, u^*_{\lambda,y}(\xi,\eta))\right| d\eta \\ &\quad + \int_0^x d\xi \int_0^y \left| f(\xi,\eta;\, u^*_\lambda(\xi,\eta);\, u^*_{\lambda,x}(\xi,\eta),\, u^*_{\lambda,y}(\xi,\eta)) \right. \\ &\qquad \left. - g^*_\lambda(\xi,\eta;\, u^*_\lambda(\xi,\eta);\, u^*_{\lambda,x}(\xi,\eta),\, u^*_{\lambda,y}(\xi,\eta))\right|\ d\eta\end{aligned}$$

Since $\{g^*_\lambda\} \xrightarrow{\text{unif}} f$ on B', $\{u^*_\lambda\} \xrightarrow{\text{unif}} u$ on R, given $\epsilon > 0$ and $(x,y) \in R$, there exists a positive integer N_1, depending upon ϵ alone, such that for $\lambda > N_1$,

$$(2.29) \qquad |u(x,y) - u^*_\lambda(x,y)| < \epsilon \ ,$$

$$(2.30) \quad \int_0^x d\xi \int_0^y |f(\xi,\eta;\ u^*_\lambda(\xi,\eta);\ u^*_{\lambda,x}(\xi,\eta),\ u^*_{\lambda,y}(\xi,\eta))$$

$$- g^*_\lambda(\xi,\eta;\ u^*_\lambda(\xi,\eta);\ u^*_{\lambda,x}(\xi,\eta),\ u^*_{\lambda,y}(\xi,\eta))|\ d\eta$$

$$< \epsilon \ell_1 \ell_2 \ .$$

Moreover, since f is uniformly continuous in B' while $\{u^*_\lambda\}$, $\{u^*_{\lambda,x}\}$, $\{u^*_{\lambda,y}\}$ converge uniformly on R to u, u_x, u_y respectively, there exists a positive integer N_2, depending on ϵ alone, such that for $\lambda > N_2$,

$$(2.31) \quad \int_0^x d\xi \int_0^y |f(\xi,\eta;\ u(\xi,\eta);\ u_x(\xi,\eta),\ u_y(\xi,\eta))$$

$$- f(\xi,\eta;\ u^*_\lambda(\xi,\eta);\ u^*_{\lambda,x}(\xi,\eta),\ u^*_{\lambda,y}(\xi,\eta))|\ d\eta$$

$$< \epsilon \ell_1 \ell_2 \ .$$

Introducing (2.29), (2.30) and (2.31) into (2.28), we obtain that for $\lambda > \max(N_1, N_2)$

$$(2.32) \quad |u(x,y) - \int_0^x d\xi \int_0^y f(\xi,\eta;\ u(\xi,\eta);\ u_x(\xi,\eta),\ u_y(\xi,\eta))$$

$$< \epsilon(1 + 2\ell_1 \ell_2)$$

But ϵ is arbitrary, hence (2.27) is verified for each $(x,y) \in R$.

We must verify the one additional fact that for each $(x,y) \in R$, $(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)) \in B$, instead of just belonging to B'.

By differentiation from (2.27),

$$(2.33)\qquad u_x(x,y) = \int_0^y f(x,\eta;\ u(x,\eta);\ u_x(x,\eta),\ u_y(x,\eta))\,d\eta$$

$$(2.34)\qquad u_y(x,y) = \int_0^x f(\xi,y;\ u(\xi,y);\ u_x(\xi,y),\ u_y(\xi,y))\,d\xi\,.$$

Hence, from the extended definition of f, (2.1), and hypothesis 3),

$$(2.35)\qquad |u(x,y)| \leq \int_0^x d\xi \int_0^y |f(\xi,\eta;\ u(\xi,\eta);\ u_x(\xi,\eta),\ u_y(\xi,\eta))|\,d\eta$$

$$\leq M\ell_1\ell_2 \leq a$$

$$(2.36)\qquad |u_x(x,y)| \leq \int_0^y |f(x,\eta;\ u(x,\eta);\ u_x(x,\eta),\ u_y(x,\eta))|\,d\eta$$

$$\leq M\ell_2 \leq b_1$$

$$(2.37)\qquad |u_y(x,y)| \leq \int_0^x |f(\xi,y;\ u(\xi,y);\ u_x(\xi,y),\ u_y(\xi,y))|\,d\xi$$

$$\leq M\ell_1 \leq b_2\ ,$$

thus completing the proof of Theorem 1a.

Remarks a) and b) to Theorem 1 apply as well to Theorem 1a.

By the following example we show that the integral surfaces for Theorem 1a are not necessarily unique:

Example 1 Consider the characteristic initial value problem:

$$(2.38)\qquad u_{xy} = |u|^{\frac{1}{2}}\ ;\quad u(x,0) = u(0,y) = 0.$$

Here $f(x,y;\ u;\ p,q) = |u|^{\frac{1}{2}}$ is continuous for all u but fails to satisfy a Lipschitz condition on u at $u = 0$. Theorem 1a applies

to insure existence of a solution in a sufficiently small neighborhood of the origin. However, at least two solutions, valid for all (x,y) in the finite plane, are directly available. First, $u \equiv 0$ obviously satisfies. Second, if we seek a solution u satisfying

i) $u \geqslant 0$,

ii) there exist functions X, Y such that
$u(x,y) = X(x) \cdot Y(y)$;

that is, by the method of separation of variables, we obtain immediately the solution $u(x,y) = \frac{1}{16} x^2 y^2$.

For purposes of illustrating the various situations that might occur, we give the following:

Example 2. Consider the characteristic initial value problem:

$$(2.39) \qquad u_{xy} = |u_x|^{\frac{1}{2}} \; ; \; u(x,0) = u(0,y) \; ; \; 0.$$

Here $f(x,y;\, u;\, p,q) = |p|^{\frac{1}{2}}$ is continuous for all p but fails to satisfy a Lipschitz condition on p at $p = 0$. Since $p(x,0) = u_x(x,0) = 0$ neither Theorem 1 nor Theorem 1a will insure existence of a solution over any domain including a portion of the x axis. Such solutions do exist, however. One is $u \equiv 0$. Under the assumption $p = u_x \geqslant 0$ we obtain another, for now

$$p_y = p^{\frac{1}{2}} \quad \text{or}$$

$$\frac{dp}{p^{\frac{1}{2}}} = 2p^{\frac{1}{2}} = y + c_1.$$

Since $p(x,0) = 0$, $c_1 = 0$ and

$$p = u_x = \frac{y^2}{4} \qquad \text{or, integrating,}$$

$$u = \frac{xy^2}{4} + c_2.$$

Since $u(0,y) = 0$, $c_2 = 0$; and hence

$$u = \frac{xy^2}{4}$$

is a second solution valid throughout the finite plane.

In Example 2 consider the function

$$u_0(x,y) = \begin{cases} 0 & \text{for} \quad x \leqslant 0 \\ \frac{xy^2}{4} & \text{for} \quad x \geqslant 0 \,. \end{cases}$$

u_0 is continuous for all (x,y) and satisfies the initial value problem (2.39) everywhere except along the y axis, where for $y \neq 0$, $u_{0x}(0,y)$ does not exist. Roughly speaking, u_0 is a continuous integral surface of problem (2.39) having a jump in the normal first derivative across a characteristic.

For equations meeting the continuity, boundedness and partial Lipschitz requirements of Theorem 1a we cannot match integral surfaces in the above fashion to obtain first derivative jumps across characteristics. This follows from the fact that if we prescribe $u(a,y) = V(y) \in C'([0,\ell_2])$ along the characteristic $x=a$, $a \in [0,\ell_1]$, then

$$(2.40) \quad \begin{cases} p_y(a,y) = f(a,y;\, V(y);\, p(a,y),\, V'(y)) \\ p(a,0) = 0 \end{cases}$$

represents a first order ordinary differential equation for the

unknown function $p = u_x$ under a one point boundary condition. The conditions that f be continuous, bounded and partially Lipschitzian are sufficient to insure the existence of a unique determination of $u_x(a,y)$ for $y \in [0,\ell_2]$. Note that in Example 2 the function f was continuous only and hence the determination of u_x from the above ordinary differential equation was not unique, thus admitting the possibility of a jump in u_x. The conditions for the determination of u_y along a characteristic y = const. are similar.

The above remarks serve to permit the extension of the domain of existence of the integral surfaces of Theorems 1 and 1a from R to R^*: $\begin{cases} -\ell_1 \leq x \leq \ell_1 \\ -\ell_2 \leq y \leq \ell_2 \end{cases}$. The arguments for the existence may be made applicable to other quadrants than the first by mere coordinate reflections. Moreover the integrals obtained in the separate quadrants must have first order contacts with each other along the coordinate axes by the above reasoning from ordinary differential equation theory. Hence we may obtain existence and uniqueness over the domain R^* by replacing B by B^*:

$$\begin{cases} -\ell_1 \leq x \leq \ell_1 \\ -\ell_2 \leq y \leq \ell_2 \\ -a \leq u \leq a \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{cases}$$

in Theorem 1; and we obtain simply existence over R^* by replacing B by B^* in Theorem 1a.

In the classical existence theorem for the ordinary differential equation $\frac{dy}{dx} = f(x,y)$, with $y(0) = 0$, the conditions that f

be continuous on $C: \begin{cases} 0 \leq x \leq a \\ -b \leq y \leq b \end{cases}$, with $M = \max|f|$ on C, were shown to be sufficient to insure existence of at least one integral curve $y = y(x)$ for $x \in [0, \alpha]$ with $\alpha \leq \min(a, \frac{b}{M})$. This bound, $\alpha \leq \min(a, \frac{b}{M})$, was shown by A. WINTNER [15] to be a maximal bound in a certain sense. We apply his method to Theorem 1a in the proof of the following:

Theorem 2.

If, in Theorem 1a, we replace B by $B'': \begin{cases} 0 \leq x \leq \ell_1' \\ 0 \leq y \leq \ell_2' \\ -\infty < u < \infty \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{cases}$

and require that f be bounded thereon, then hypothesis 3) in that theorem reduces to

3)' $\ell_1 \leq \min(\ell_1', \frac{b_2}{M})$, $\ell_2 \leq \min(\ell_2', \frac{b_1}{M})$,

where $M = \max|f|$ on B''. Moreover, the bounds established by 3)' are maximal bounds in a sense to be explained below.

Proof.

The condition $M \ell_1 \ell_2 \leq a$ of hypothesis 3) is immediately satisfied since a approaches $+\infty$. The conditions $M \ell_1 \leq b_2$, $M \ell_2 \leq b_1$ may be rewritten as in 3)' and are now the only restrictions on ℓ_1 and ℓ_2.

If $\ell_1' \leq \frac{b_2}{M}$, $(\ell_2' \leq \frac{b_1}{M})$, then the conclusion is immediate. For, we may take $f(x,y;\, u;\, p,q) = h(x)$, $(g(y))$, which function is not even defined for $x > \ell_1 = \ell_1'$, $(y > \ell_2 = \ell_2')$.

Suppose $\ell_2' > \frac{b_1}{M}$. Then we consider the sequence of problems:

$$u_{xy} = (2^{1/m} - u_x)^{1/m+1}, \quad u(x,0) = u(o,y) = 0, \quad (m=1,2,\cdots). \tag{2.41}$$

Setting $p = u_x$, (2.41) becomes

$$p_y(x,y) = (2^{1/m} - p(x,y))^{1/m+1}, \quad p(x,0) = 0.$$

Integrating this ordinary differential equation for p as a function of y, we obtain

$$p(x,y) = 2^{1/m} - \left[2^{1/m+1} - \frac{m}{m+1}\,y\right]^{m+1/m}.$$

But, since $p = u_x$ and $u(0,y) = 0$ we may integrate again to obtain

$$u(x,y) = x\left\{2^{1/m} - \left[\frac{m}{m+1}(C_m - y)\right]^{m+1/m}\right\} \tag{2.42}$$

where

$$C_m = \frac{m+1}{m}\, 2^{\frac{1}{m+1}}. \tag{2.43}$$

The line $y = C_m$ is a branch line of the solution u. Under the supposition $\ell_2' > \frac{b_1}{M}$, the desired statement is that $\frac{b_1}{M}$ is a maximal bound on ℓ_2, i.e., for each $\epsilon > 0$, there exists a function $f(x,y;\, u;\, p,q)$, depending on ϵ and satisfying hypotheses 1), 2)$'$ and 3)$'$ on B", such that an integral $u(x,y)$ of the problem corresponding to f has a singularity for some $y \in (\frac{b_1}{M}, \frac{b_1}{M} + \epsilon)$.

Defining

$$f_m(x,y;\, u;\, p,q) = (2^{1/m} - p)^{1/m+1} \text{ for } -2^{1/m+1} \leq p \leq 2^{1/m+1},$$

$(m = 1,2,\cdots)$, we obtain

$b_{1m} = 2^{1/m+1}$, $M_m = (2^{1/m} + 2^{1/m+1})^{1/m+1}$; and, since

$$(2^{1/m} + 2^{1/m+1}) > 2, \ (m = 1,2,\cdots),$$

$$\lim_{m \to \infty} \frac{b_{1m}}{M_m} = 1 - .$$

Moreover, by (2.43),

$$\lim_{m \to \infty} C_m = 1 \quad .$$

Hence, given $\epsilon > 0$, there exists a positive integer M, depending on ϵ alone, such that $m > M \implies$

$$\frac{b_{1m}}{M_m} + \epsilon > C_m > \frac{b_{1m}}{M_m} \quad .$$

Consequently $\dfrac{b_1}{M}$ is a maximal bound on ℓ_2.

To determine that the condition $\ell_1 \leq \min\,(\ell_1', \frac{b_2}{M})$ is also a maximal bound we consider the sequence of problems.

$$(2.44) \quad u_{xy} = (2^{1/m} - u_y)^{1/m+1}, \ u(x,0) = u(0,y), \ (m = 1,2,\cdots),$$

and follow the same line of reasoning as before. Thus Theorem 2 is verified.

The close parallelism between our conclusions and the classical theorems for first order ordinary differential equations

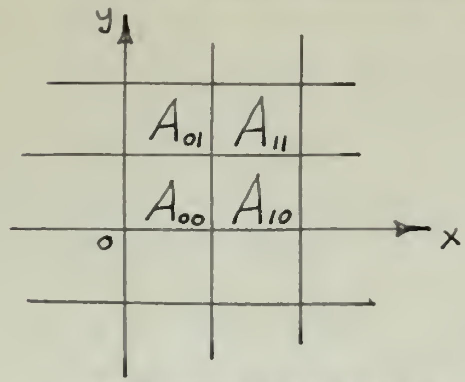
(See E. KAMKE [2]) leads one to suspect that an existence theorem might be proved wherein mere continuity of the function f was demanded. The analogue to the Cauchy polygon method is the attack suggested by the parallelism, and it leads to an existence theorem for the characteristic initial value problem:

(2.45) $$u_{xy} = f(x,y;\, u) \quad , \quad u(x,0) = u(0,y) = 0.$$

We do not give the proof here; first, because the theorem is a special case of Theorem 1a; and, second, because the steps in the proof are practically identical with those of the Cauchy polygon method for ordinary differential equations.

When $f = f(x,y;\, u;\, p,q)$ and f is merely continuous this attack involves difficulties which we have not been able to resolve. We sketch the method to indicate the source of trouble:

In a neighborhood of the origin a partition π by characteristics is specified where the subregions A_{ij} in the first quadrant are defined as



$$A_{ij}: \begin{cases} x_i \leq x < x_{i\;1} \\ y_j \leq y < y_{j\;1} \end{cases} \quad (i,j=0,1,2\cdots)$$

We formulate the approximate integral surface u corresponding to the partition π as follows:

(2.46) $$u_\pi(x,y) = \int_0^x d\xi \int_0^y F_\pi(\xi,\eta)\, d\eta,$$

where

$$F_\pi(x,y) = f(x_i, y_j; u_\pi(x_i,y_j); u_{\pi_x}(x_i,y_j), u_{\pi_y}(x_i y_j)) \tag{2.47}$$

for $(x,y) \in A_{ij}$.

The principal difficulty lies in the fact that the derivatives

$$u_{\pi_x} = \int_0^y F_\pi(x,\eta)\, d\eta \quad \text{and} \tag{2.48}$$

$$u_{\pi_y} = \int_0^x F_\pi(\xi,y)\, d\xi \tag{2.49}$$

are discontinuous across the partition lines x = constant and y = constant, respectively, thus preventing the direct application of ARZELA's theorem on equicontinuous functions when we consider the sequence of approximate integral surfaces formed by partition refinement.

The equation of (2.45) appears to be more amenable than the more general equation involving the first derivatives p and q. G. FUBINI [16] p. 622, by demanding only that $f(x,y;u)$ be continuous and Lipschitzian with respect to u, has proved the existence of a unique integral of $u_{xy} = f(x,y;u)$ satisfying Dirichlet conditions, i.e. the value of u prescribed on a closed contour. This result, while remarkable, is not contradictory since u is shown to have a discontinuity of the second type at one point of the boundary.

We conclude this chapter with the statement of the extension of Theorems 1 and 1a to a system of equations

$$s_i = f_i(x,y;\ u_1,\cdots,u_n;\ p_1,\cdots p_n, q_1,\cdots,q_n),\ (i=1,2,\cdots,n) \tag{2.50}$$

satisfying the initial conditions

$$u_i(x,0) = u_i(0,y) = 0\ ,\quad (i=1,2,\cdots,n). \tag{2.51}$$

Theorem 3, below, is a natural extension of Theorem 1. In principle, it was first obtained by O. NICCOLETTI [14] p.7. His statement, however, is not explicit about the bounds on the domain of existence. Moreover, to prove uniqueness he requires the f_i to be of class C'. We obtain the improved statement, Theorem 3, by modifying the arguments of E. KAMKE [2] p. 402 and p. 408 to apply them to the system (2.50).

Theorem 3)

1) $f_i(x,y;\ u_j;\ p_j,q_j)$[2] $\in C(B'')$, B'': $\begin{cases} 0 \le x \le \ell_1 \\ 0 \le y \le \ell_2 \\ -a \le u_i \le a \\ -b_1 \le p_i \le b_1 \\ -b_2 \le q_i \le b_2 \end{cases}$

2) The f_i are Lipschitzian on B''; i.e. there exists a positive constant K such that for $(x,y;\ u^1{}_j;\ p^1{}_j,\ q^1{}_j) \in B''$, $(x,y;\ u^2{}_j;\ p^2{}_j,q^2{}_j) \in B''$, and $i = 1,2,\cdots,n$,

$$|f_i(x,y;\ u^1{}_j;\ p^1{}_j,q^1{}_j) - f_i(x,y;\ u^2{}_j;\ p^2{}_j,q^2{}_j)|$$
$$\le K \sum_{j=1}^{n} \left\{ |u^1{}_j - u^2{}_j| + |p^1{}_j - p^2{}_j| + |q^1{}_j - q^2{}_j| \right\}.$$

3) $M\ell_1\ell_2 \le a$, $M\ell_1 \le b_2$, $M\ell_2 \le b_1$ where $M = \max\left\{ |f_1|,\cdots,|f_n| \right\}$ on B''.

---

[2] Notation: $(x,y;\ u_j;\ p_j,q_j) = (x,y;\ u_1,\cdots,u_n;p_1,\cdots,p_n, q_1,\cdots,q_n)$.

$\Longrightarrow$ 4) There exists one and only one set of functions $\{u_1,\cdots,u_n\}$, $u_j(x,y) \in C'(R)$, $u_{j,xy}(x,y) \in C(R)$, $(j=1,\cdots,n)$, where $R:\begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$, such that for each $(x,y) \in R$ the point $(x,y;u_j(x,y);\, u_{j,x}(x,y),\, u_{j,y}(x,y)) \in B''$, and

$$u_{i,xy}(x,y) = f_i(x,y;u_j(x,y);\, u_{j,x}(x,y),\, u_{j,y}(x,y)),$$

$$u_i(x,0) = u_i(0,y) = 0, \quad (i = 1,\cdots,n), \text{ for each } (x,y) \in R.$$

By relaxing hypothesis 2) we obtain the improved theorem below; which stands in the same relation to Theorem 3 that Theorem 1a does to Theorem 1.

Theorem 3a

1)

2)' The f_i are partially Lipschitzian on B''; i.e. there exists a positive constant K such that for $(x,y;\, u_j;\, p^1_j,\, q^1_j) \in B''$, $(x,y;u_j;p^2{}_j,q^2{}_j) \in B''$, and $i = 1, 2,\cdots, n$,

$$\left| f_i(x,y;\, u_j;\, p^1{}_j,\, q^1{}_j) - f_i(x,y;\, u_j;\, p^2{}_j,\, q^2{}_j) \right| \leq K \sum_{j=1}^{n} \left\{ |p^1{}_j - p^2{}_j| + |q^1{}_j - q^2{}_j| \right\}.$$

3)

$\Longrightarrow$ 4)' There exists at least one set of functions $\{u_1,\cdots,u_n\}$, $u_j(x,y) \in C'(R)$, $u_{j,xy}(x,y) \in C(R)$, $(j=1,\cdots,n)$, where

$R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$, such that for each $(x,y) \in R$ the point

$(x,y;\ u_j(x,y);\ u_{j,x}(x,y),\ u_{j,y}(x,y)) \in B''$, and

$$u_{i,xy}(x,y) = f_i(x,y;\ u_j(x,y);\ u_{j,x}(x,y),\ u_{j,y}(x,y)),$$

$$u_i(x,0) = u_i(0,y) = 0, \quad (i = 1,\cdots n), \text{ for each } (x,y) \in R.$$

The proof of Theorem 3a is essentially a step by step repetition of that for Theorem 1a. WEIERSTRASS' theorem tells us that for each positive integer i there exists a sequence of polynomials $\{g_{i\lambda}\}$ $(x,y;\ u_j;\ p_j,q_j)$, $(\lambda = 1,2,\cdots)$, converging uniformly on B'' to $f_i(x,y;\ u_j;\ p_j,q_j)$. We extend the $g_{i\lambda}$ and the f_i as before and obtain that there exist positive constants L_i such that for each i $|g_{i\lambda}| \leq L_i$ on B'', extended, and for all λ. We let $L = \max\{L_1,\cdots, L_n\}$ and proceed as before, using Theorem 3 in place of Theorem 1 to obtain the integral $u_{i\lambda}$ associated with each $g_{i\lambda}$.

We note only one point of significant difference in the arguments. In place of inequality (2.13) of Lemma 2 we now have the inequalities

$$|u_{i\lambda,x}(x_2,y) - u_{i\lambda,x}(x_1,y)|$$
$$\leq K\int_0^y \left\{ \sum_{j=1}^{n} |u_{j\lambda,x}(x_2,\eta) - u_{j\lambda,x}(x_1,\eta)| \right\} d\eta \qquad (i=1,\ldots,n)$$

Summing these, and letting

$$Z(y) = \sum_{i=1}^{n} |u_{i\lambda,x}(x_2,y) - u_{i\lambda,x}(x_1,y)|,$$

we obtain

$$0 \leq Z(y) \leq Kn \int_0^y Z(\eta)\, d\eta + n(\mu + \zeta)$$

to which Lemma 1 applies. Thus the equicontinuity of each of the sequences $\left\{ u_{i\lambda,x} \right\}$, $(i = 1,\cdots,n)$ is assured.

Remarks a) and b) to Theorems 1 and 1a apply, with obvious modifications, to Theorems 3 and 3a. Moreover, as before, we may extend the domain of existence of the integral surfaces of Theorems 3 and 3a from R to R*: $\begin{cases} -\ell_1 \leq x \leq \ell_1 \\ -\ell_2 \leq y \leq \ell_2 \end{cases}$.

The set of functions $\left\{ u_1, \cdots, u_n \right\}$ representing the solution to the problem of Theorem 3a cannot be shown to be unique. This is made evident by extending Example 1 to the system

$$\begin{array}{ll} u_{1,xy} = |u_1|^{\frac{1}{2}}, & u_1(x,0) = u_1(0,y) = 0 \\ u_{2,xy} = 0, & u_2(x,0) = u_2(0,y) = 0 \\ \vdots & \vdots \qquad\qquad \vdots \\ u_{n,xy} = 0, & u_n(x,0) = u_n(0,y) = 0 \end{array}$$

for which $u_i \equiv 0 \qquad (i = 2,\cdots,n)$
while $u_1 \equiv 0$ or $u_1 = \frac{1}{16} x^2 y^2$. Thus at least two sets of solutions are possible for this system which satisfies the hypotheses of Theorem 3a.

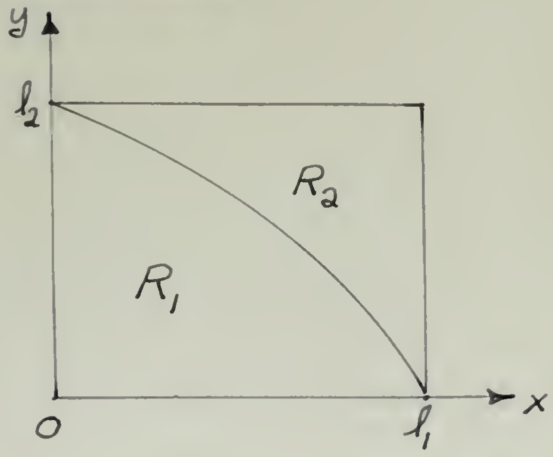
CHAPTER III

The Cauchy Problem for $u_{xy} = f(x,y;\ u;\ u_x,u_y)$.

The development of this chapter closely parallels that of Chapter II. Consequently, the notation will be abridged, in particular with respect to the arguments of functions; and the proofs will be merely outlined to show minor variations from the statements in Chapter 2.

For reference, we state the following theorem proved first for systems of equations by O: NICCOLETTI [14] p. 7. Our statement may be easily inferred from that of E. KAMKE [2] p. 405 and p. 410, by a slight modification of his proof.

Theorem 4



1) $f(x,y;u;p,q) \in C(B)$,

$$B: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \\ -a \leq u \leq a \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{cases}$$

2) f is Lipschitzian on B, (as defined in Theorem 1).

3) $M\,\ell_1\,\ell_2 \leq a$, $M\,\ell_1 \leq b_2$, $M\,\ell_2 \leq b_1$, where $M = \max |f|$ on B

4) $\Upsilon: \begin{cases} 0 \leq x \leq \ell_1 \\ y = \varphi(x) \end{cases}$ where $\varphi(x) \in C'([0,\ell_1])$, $\varphi'(x) \neq 0$ for $x \in [0,\ell_1]$ and $\varphi(0) = \ell_2$, $\varphi(\ell_1) = 0$.

$\Longrightarrow$ 5) There exists one and only one function $u(x,y) \in C'(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$, such that for each $(x,y) \in R$, the point $(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)) \in B$, and

$$u_{xy}(x,y) = f(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)),$$

$$u(x,\varphi(x)) = u_x(x,\varphi(x)) = u_y(x,\varphi(x)) = 0$$

for each $(x,y) \in R$.

Remarks c) Suppose we prescribe $u(x,\varphi(x)) = U(x)$, $u_x(x,\varphi(x)) = P(x)$, $u_y(x,\varphi(x)) = Q(x)$ where $U(x) \in C'([0,\ell_1])$ while $P(x), Q(x) \in C([0,\ell_1])$. Our prescription must satisfy the strip condition $U' = P + Q \cdot \varphi'$ for each $x \in [0,\ell_1]$. Consider the function $w(x,y) = U(x) + (y - \varphi(x))\,Q(x)$. Clearly, $w_{xy} = Q'(x)$ while $w(x,\varphi(x)) = U(x)$, $w_x(x,\varphi(x)) = P(x)$, and $w_y(x,\varphi(x)) = Q(x)$. Hence the function $v = u - w$ must satisfy $v_{xy} = Q'(x) + f(x,y;\ v + w;\ v_x + w_x,\ v_y + w_y)$, with $v(x,\varphi(x)) = v_x(x,\varphi(x)) = v_y(x,\varphi(x)) = 0$, a problem of the type covered by Theorem 4.

d) Hypothesis 4) of Theorem 4 is more restrictive than it need be. At isolated points of Υ we may have a horizontal or vertical tangent, provided that Υ does not cross the same characteristic more than once. For, under these conditions the inverse function ψ to φ will exist and be continuous for all $y \in [0,\ell_2]$.

Our improvement of this theorem is as follows:

Theorem 4a

1)

2)' f is partially Lipschitzian on B, (as defined in Theorem 1a).

3)

4)

$\Longrightarrow$ 5) There exists at least one function $u(x,y) \in C'(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leqslant x \leqslant \ell_1 \\ 0 \leqslant y \leqslant \ell_2 \end{cases}$, such that for each $(x,y) \in R$, the point $(x,y;\, u(x,y);\, u_x(x,y),\, u_y(x,y)) \in B$, and

$$u_{xy}(x,y) = f(x,y;\, u(x,y);\, u_x(x,y),\, u_y(x,y)),$$

$$u(x,\varphi(x)) = u_x(x,\varphi(x)) = u_y(x,\varphi(x)) = 0$$

for each $(x,y) \in R$.

Outline of proof.

The path Υ may also be expressed as $\Upsilon: \begin{cases} x = \psi(y) \\ 0 \leqslant y \leqslant \ell_2 \end{cases}$ where $\psi(y) \in C'([0,\ell_2])$, $\psi'(y) \neq 0$ for $y \in [0,\ell_2]$. ψ is the inverse function to φ.

We may express the problem as the integral equation

(3.1)
$$u(x,y) = \int_{\psi(y)}^{x} d\xi \int_{\varphi(\xi)}^{y} f(\xi,\eta;u;u_x,u_y)\, d\eta$$

$$= \int_{\varphi(x)}^{y} d\eta \int_{\psi(\eta)}^{x} f(\xi,\eta;u;u_x,u_y)\, d\xi$$

Whence

(3.2) $$u_x(x,y) = \int_{\varphi(x)}^{y} f(x,\eta;u;u_x,u_y)\, d\eta$$

(3.3) $$u_y(x,y) = \int_{\psi(y)}^{x} f(\xi,y;\, u;\, u_x,u_y)\, d\xi\,.$$

By WEIERSTRASS' theorem, there exists a sequence of polynomials $\{g_\lambda\} \xrightarrow{\text{unif.}} f$ on B. We extend the domain of definition of f and the polynomials g_λ over B to B' by definition (2.1).

We obtain again the constant $L > 0$ such that $|g_\lambda| \leqslant L$ in B' for all λ. Moreover, for each g_λ the Lipschitz condition (2.2) is satisfied. Thus, by Theorem 4, for each λ there exists a unique solution u_λ to the problem

$$(3.4) \left\{ \begin{array}{l} u_{\lambda,xy} = g_\lambda(x,y;\ u_\lambda;\ u_{\lambda,x},\ u_{\lambda,y}), \\ u_\lambda(x,\varphi(x)) = u_{\lambda,x}(x,\varphi(x)) = u_{\lambda,y}(x,\varphi(x)) = 0. \end{array} \right.$$

That the sequences $\{u_\lambda\}$, $\{u_{\lambda,x}\}$, $\{u_{\lambda,y}\}$ are uniformly bounded on R, and that the sequence $\{u_\lambda\}$ is equicontinuous on R is immediately evident from the equivalent integral expressions

$$(3.5) \quad u_\lambda(x,y) = \int_{\psi(y)}^{x} d\xi \int_{\varphi(\xi)}^{y} g_\lambda(\xi,\eta;\ u_\lambda;\ u_{\lambda,x},u_{\lambda,y})\, d\eta$$

$$= \int_{\varphi(x)}^{y} d\eta \int_{\psi(\eta)}^{x} g_\lambda(\xi,\eta; u_\lambda; u_{\lambda,x},u_{\lambda,y})\, d\xi,$$

$$(3.6) \quad u_{\lambda,x}(x,y) = \int_{\varphi(x)}^{y} g_\lambda(x,\eta; u_\lambda; u_{\lambda,x},u_{\lambda,y})\, d\eta,$$

$$(3.7) \quad u_{\lambda,y}(x,y) = \int_{\psi(y)}^{x} g_\lambda(\xi,y;\ u_\lambda;\ u_{\lambda,x},u_{\lambda,y})\, d\xi .$$

We now establish the equicontinuity of $\{u_{\lambda x}\}$ and of $\{u_{\lambda y}\}$. This done, the same arguments as those for the proof of Theorem 1a will serve to obtain a subsequence $\{u_{\lambda^*}\}$ of $\{u_\lambda\}$ which converges uniformly to the solution u.

There is no loss in generality in restricting ourselves at this point to the consideration of those points $(x,y) \in R_2 : \begin{cases} 0 \leq x \leq \ell_1 \\ \varphi(x) \leq y \leq \ell_2 \end{cases}$. For we shall see that the arguments developed below will apply as well for $(x,y) \in R_1 : \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \varphi(x) \end{cases}$ after a simple coordinate translation and rotation. Thus if we insure existence of a solution on R_2, existence on R_1 is simultaneously verified. Moreover, the Cauchy initial data insure that such integral surfaces have a first order contact along Υ and hence define an integral surface throughout all of $R = R_1 + R_2$.

Given points $(x_2, y_2) \in R_2$, $(x_1, y_1) \in R_2$, it is always possible to label these points in such a way that $(x_1, y_2) \in R_2$. This being done, we have that

$$(3.8) \quad \left| u_{\lambda,x}(x_1,y_2) - u_{\lambda,x}(x_1,y_1) \right| \leq L \left| y_2 - y_1 \right| ,$$

$$(3.9) \quad \left| u_{\lambda,y}(x_2,y_2) - u_{\lambda,y}(x_1,y_2) \right| \leq L \left| x_2 - x_1 \right| .$$

Assuming, without loss, that $y \geqslant \varphi(x_2) \geqslant \varphi(x_1)$, we have that

$$(3.10) \quad u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y) = \int_{\varphi(x_2)}^{y} \Big[g_\lambda(x_2,\eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) - g_\lambda(x_1,\eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) \Big] d\eta + \int_{\varphi(x_1)}^{\varphi(x_2)} g_\lambda(x_1,\eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\eta$$

We operate on the first integral on the right hand side of (3.10) in the manner demonstrated in equation (2.20). We obtain a formula identical with (2.20) except that here the lower limit of integration is $y = \varphi(x_2)$ instead of $y = 0$. For brevity, we omit the formula.

Since

$$(3.11)\quad \left|\int_{\varphi(x_1)}^{\varphi(x_2)} g_\lambda(x_1,\eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y})\,d\eta\right| \leq L\left|\varphi(x_2)-\varphi(x_1)\right|, \; (\lambda=1,2,\cdots)$$

and since $\varphi(x)$ is uniformly continuous on $[0,\ell_1]$, by the same reasoning as before we arrive at the slight modification to Lemma 2,

$$(3.12)\quad \left|u_{\lambda,x}(x_2,y)-u_{\lambda,x}(x_1,y)\right| \leq K\int_{\varphi(x_2)}^{y}\left|u_{\lambda,x}(x_2,\eta)-u_{\lambda,x}(x_1,\eta)\right|d\eta + \mu + \zeta$$

from which, by Lemma 1,

$$(3.13)\quad \left|u_{\lambda,x}(x_2,y)-u_{\lambda,x}(x_1,y)\right| \leq (\mu+\zeta)e^{k(y-\varphi(x_2))} \leq (\mu+\zeta)e^{k\ell_2} .$$

The equicontinuity of $\{u_{\lambda,x}\}$ is thus assured.

The argument for the equicontinuity of $\{u_{\lambda,y}\}$ is similar, hence Theorem 4a obtains.

Remarks c) and d) to Theorem 4 apply as well to Theorem 4a. Quite obviously, if f is continuous, bounded and Lipschitzian (or partially Lipschitzian) on the infinite cylinder with cross section R, then hypothesis 3) of Theorem 4 (or 4a) is immediately satisfied. In fact, this was the form of Theorem 4 which was utilized in the proof of Theorem 4a.

As previously mentioned, the extension of Theorem 4 to systems of equations was first obtained, in principle, by O. NICCOLETTI [14]. He was not, however, explicit about the domain of existence of the solution. The following statement may be derived

from the same arguments of E. KAMKE [2] p. 405 and p. 410 used as the basis for Theorem 4.

Theorem 5.

1) $f_i(x,y;\, u_1, \cdots, u_n;\, p_1, \cdots, p_n,\, q_1, \cdots, q_n) \in C(B^*)$

$$B^*: \begin{cases} 0 \le x \le \ell_1 \\ 0 \le y \le \ell_2 \\ -a \le u_i \le a \\ -b_1 \le p_i \le b_1 \\ -b_2 \le q_i \le b_2 \end{cases} \qquad (i = 1, \cdots, n).$$

2) The f_i are Lipschitzian on B^*, (as defined in Theorem 3).

3) $M \ell_1 \ell_2 \le a$, $M \ell_1 \le b_2$, $M \ell_2 \le b_1$, where

$$M = \max \left\{ |f_1|, \cdots, |f_n| \right\} \text{ on } B^*.$$

4) $\Upsilon : \begin{cases} 0 \le x \le \ell_1 \\ y = \varphi(x) \end{cases}$ where $\varphi(x) \in C'([0,\ell_1])$, $\varphi'(x) \ne 0$

for $x \in [0,\ell_1]$ and $\varphi(0) = \ell_2$, $\varphi(\ell_1) = 0$.

$\Longrightarrow$ 5) There exists one and only one set of functions $\{u_1, \cdots, u_n\}$, $u_i(x,y) \in C'(R)$, $u_{i,xy}(x,y) \in C(R)$, $(i = 1, \cdots, n)$, where $R: \begin{cases} 0 \le x \le \ell_1 \\ 0 \le y \le \ell_2 \end{cases}$, such that for each $(x,y) \in R$ the point $(x,y;\, u_j(x,y);\, u_{j,x}(x,y),\, u_{j,y}(x,y)) \in B$, and

$$u_{i,xy}(x,y) = f_i(x,y;\, u_j(x,y),\, u_{j,x}(x,y),\, u_{j,y}(x,y)),$$

$$u_i(x, \varphi(x)) = u_{i,x}(x, \varphi(x)) = u_{i,y}(x, \varphi(x)) = 0,$$

$(i = 1, \cdots, n)$, for each $(x,y) \in R$.

We may extend the arguments in the proof of Theorem 4a to apply to systems of equations. The extension is practically identical with the previous extension of Theorem 1a to Theorem 3a, except that now Theorem 5 is used to establish existence and uniqueness of the solutions of the system

$$u_{i\lambda,xy} = g_{i\lambda}(x,y;\, u_{j\lambda};\, u_{j\lambda,x},\, u_{j\lambda,y}),\ (i=1,\cdots,n),$$

$$(\lambda = 1,2,\cdots),$$

under the Cauchy initial conditions. We obtain the following theorem:

Theorem 5a

1)

2)' the f_i are partially Lipschitzian on B'', (as defined in Theorem 3a).

3)

4)

⟹ 5)' There exists at least one set of functions $\{u_1,\cdots,u_n\}$, $u_i(x,y) \in C'(R)$, $u_{i,xy}(x,y) \in C(R)$, $(i = 1,\cdots,n)$, where $R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$, such that for each $(x,y) \in R$ the point $(x,y;\, u_j(x,y);\, u_{j,x}(x,y),\, u_{j,y}(x,y)) \in B$, and

$u_{i,xy}(x,y) = f_i(x,y;\, u_j(x,y);\, u_{j,x}(x,y),\, u_{j,y}(x,y))$,

$u_i(x,\varphi(x)) = u_{i,x}(x,\varphi(x)) = u_{i,y}(x,\varphi(x)) = 0$,

$(i = 1,\cdots, n)$, for each $(x,y) \in R$.

Remark c), with obvious modifications, and Remark d) to Theorem 4 apply as well as to Theorems 5 and 5a. Moreover, in Theorem 5 (or 5a) we may eliminate hypothesis 3) by demanding that the f_i be continuous, bounded and Lipschitzian (or partially Lipschitzian) on the infinite cylinder with cross section R.

CHAPTER IV

Existence Theorems for Canonical Hyperbolic First Order Systems

In this chapter, and in Chapters 5 and 6 as well, we shall not give explicit domains of definition for the functions involved in the differential equations. As a consequence, existence will be shown "in the small" only. We do this because our method of attack will not yield any improvement upon the domains of existence, no matter how large the domains of definition are taken, provided the other hypotheses are not weakened. We shall elaborate on this peculiarity in the course of the exposition.

Theorems 6 and 7 belowwere given by M. CINQUINI-CIBRARIO [12] p. 180 in the form stated. A statement under somewhat weaker hypotheses, but based on the same proof, may be found in R. COURANT-D. HILBERT [17] p. 324. We examine the proof to show that the arguments therein are independent of the uniqueness of the solutions to the problems involved. As a consequence, our results in Chapters 2 and 3 apply and we arrive at the improved statements given by Theorems 6a and 7a, where hypothesis 2) of Theorems 6 and 7 is dropped altogether and the corresponding conclusions are altered to read "at least one".

Common hypothesis 1) We shall suppose the functions a_{ik}, c_i, $(i,k=1,\cdots,n)$, of arguments $x,y,u_1,\cdots,u_n$, to be continuously differentiable with bounded derivatives in a certain domain D. Fur-

ther, we suppose the determinant

$$(4.1) \qquad |a_{ik}| \neq 0 \quad \text{in } D.$$

Under these assumptions, the system

$$(4.2) \begin{cases} A_i(x,y) = \sum_{k=1}^{n} a_{ik}\, u_{k,x}(x,y) - c_i = 0, & (i=1,\cdots,m<n) \\ B_i(x,y) = \sum_{k=1}^{n} a_{ik}\, u_{k,y}(x,y) - c_i = 0, & (i=m+1,\cdots,n) \end{cases}$$

is called a canonical hyperbolic first order system.

Theorem 6. (Characteristic initial value problem.)

1)

2) All first derivatives of the functions a_{ik}, c_i, $(i,k=1,\cdots,n)$ satisfy a Lipschitz condition with respect to arguments $u_1,\cdots,u_n$ in D.

$$3) \quad \left.\begin{array}{l} U_i(x) \in C'([0,\ell_1]) \\ V_i(y) \in C'([0,\ell_2]) \\ U_i(0) = V_i(0) \end{array}\right\} \qquad (i=1,\cdots,n)$$

Moreover, for each $x \in [0,\ell_1]$, the point $(x,0;U_j(x))^3 \in D$ and

$$(4.3) \quad \sum_{k=1}^{n} a_{ik}(x,0;\, U_j(x))U'_k(x) - c_i(x,0;U_j(x)) = 0,$$
$$(i=1,\cdots,m<n);$$

and, for each $y \in [0,\ell_2]$, the point $(0,y;\, V_j(y)) \in D$ and

$$(4.4) \quad \sum_{k=1}^{n} a_{ik}(0,y;V_j(y))V'_k(y) - c_i(0,y;\, V_j(y)) = 0,$$
$$(i=m+1,\cdots,n).$$

---

[3] Recall the notation: $(x,y;U_j(x)) = (x,y;U_1(x),\cdots,U_n(x))$.

$\Longrightarrow$ 4) There exists one and only one set of functions $\left\{u_1,\cdots,u_n\right\}$, $u_i(x,y) \in C'(R_\eta)$, $u_{i,xy} \in C(R_\eta)$, $(i = 1,\cdots,n)$, where $R_\eta : \begin{cases} 0 \le x \le \eta \ell_1 \\ 0 \le y \le \eta \ell_2 \end{cases}$, with $0 < \eta \le 1$ and η sufficiently small, such that the set of functions satisfies the system (4.2) for each $(x,y) \in R_\eta$ and satisfies the conditions

$$\left.\begin{array}{ll} u_i(x,0) = U_i(x) & \text{for } x \in [0, \ell_1] \\ u_i(0,y) = V_i(y) & \text{for } y \in [0, \ell_2] \end{array}\right\} (i = 1,\cdots,n).$$

Theorem 6a.

1)

3)

$\Longrightarrow$ 4)' There exists at least one set of functions, etc. (as in Theorem 6).

Theorem 7. (Cauchy problem.)

1)

2) (as in Theorem 6.)

5) $\Upsilon : \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases}$ for $\tau \in [0,1]$, $x(\tau)$ and $y(\tau) \in C'([0,1])$ and strictly monotone, i.e., $\dot{x} \ne 0$, $\dot{y} \ne 0$ on $[0,1]$. $U_i(\tau) \in C'([0,1])$, $(i = 1,\cdots,n)$. For each $\tau \in [0,1]$, the point $(x(\tau), y(\tau); U_j(\tau)) \in D$.

$\Longrightarrow$ 6) There exists one and only one set of functions $\left\{u_1,\cdots,u_n\right\}$, $u_i(x,y) \in C'(R_\Upsilon)$, $u_{i,xy}(x,y) \in C(R_\Upsilon)$, $(i = 1,\cdots,n)$, where R_Υ is a sufficiently small neighborhood of the curve Υ, such that

the set of functions satisfies the system (4.2) for each $(x,y) \in R_{\gamma}$ and satisfies the conditions

$$u_i(x(\tau), y(\tau)) = U_i(\tau) \quad \text{for } \tau \in [0,1], \ (i = 1,\cdots,n).$$

Theorem 7a

1)

5)

$\Longrightarrow$ 6)' There exists at least one set of functions etc. (as in Theorem 7.)

The proofs of these theorems are contained in the following argument:

Suppose we have a set of functions $\{u_1,\cdots,u_n\}$, either as a solution to the characteristic initial value problem above on a domain R_{η}, or as a solution to the Cauchy problem above on a domain R_{γ}. Then for either case,

$$(4.5) \quad A_{i,y} = \sum_{k=1}^{n} a_{ik}\, u_{k,xy} + \sum_{k=1}^{n} \left[a_{ik,y} + \sum_{r=1}^{n} \frac{\partial a_{ik}}{\partial u_r}\, u_{r,y} \right] u_{k,x}$$

$$- c_{i,y} - \sum_{k=1}^{n} \frac{\partial c_i}{\partial u_k}\, u_{k,y} = 0, \ (i = 1,\cdots,m<n),$$

$$(4.6) \quad B_{i,x} = \sum_{k=1}^{n} a_{ik}\, u_{k,xy} + \sum_{k=1}^{n} \left[a_{ik,x} + \sum_{r=1}^{n} \frac{\partial a_{ik}}{\partial u_r}\, u_{r,x} \right] u_{k,y}$$

$$- c_{i,x} - \sum_{k=1}^{n} \frac{\partial c_i}{\partial u_k}\, u_{k,x} = 0, \ (i = m+1,\cdots, n).$$

Equations (4.5) and (4.6) are n linear algebraic equations in the

n unknowns $u_{i,xy}$. Since the determinant of this system, $|a_{ik}|$, does not vanish over the domain in question, we may solve the system to obtain explicitly

$$(4.7) \quad u_{i,xy} = f_i(x,y;\, u_j;\, u_{j,x},\, u_{j,y}),\ (i = 1,\cdots,n).$$

Under hypothesis 1) alone, the f_i are continuous and partially Lipschitzian over any bounded domain in the $3n + 2$ dimensional $(x,y;\, u_j;\, u_{j,x},\, u_{j,y})$-space where $(x,y;\, u_j) \in D$. If hypothesis 2) also applies, the f_i are "fully" Lipschitzian over any such domain.

Consider Theorems 6 and 6a. The characteristic initial conditions imposed therein, coupled with the system (4.7), form a problem of the type considered in Theorems 3 and 3a, respectively. (Chapter 2). We have shown above that any solution of a canonical hyperbolic system is also a solution of a particular system of type (4.7). If we now demonstrate the converse for characteristic initial conditions, i.e. that any solution of the derived system (4.7) is also a solution of the original system (4.2), then Theorems 6 and 6a follow directly from Theorems 3 and 3a respectively.

Suppose we have a set of functions $\left\{u_1,\cdots,u_n\right\}$ as a solution of (4.7) over a certain domain including the initial characteristics. By (4.5) and (4.6), which are merely alternative forms of (4.7), we havd

$$(4.8)\quad \begin{cases} A_{i,y}(x,y) = 0 \quad , \quad (i = 1,\cdots, m < n) \\ B_{i,x}(x,y) = 0 \quad , \quad (i = m+1,\cdots, n) \end{cases}$$

over this domain. But, by (4.3) and (4.4) of hypothesis 3) to both Theorems 6 and 6a, we have that

$$(4.9)\quad \begin{cases} A_i(x,0) = 0 \quad , \quad (i = 1,\cdots, m < n) \\ B_i(0,y) = 0 \quad , \quad (i = m+1,\cdots n), \end{cases}$$

whence

$$A_i(x,y) \equiv 0 \quad , \quad (i = 1,\cdots, m < n),$$
$$B_i(x,y) \equiv 0 \quad , \quad (i = m+1,\cdots, n),$$

throughout the domain. Hence the converse is shown.

For the Cauchy problem considered in Theorems 7 and 7a, we observe first that we can determine $u_{i,x}(x(\tau), y(\tau))$ and $u_{i,y}(x(\tau), y(\tau))$, $(i = 1,\cdots,n)$, as functions continuous for each $\tau \in [0,1]$, solely from the prescription of $u_i(x(\tau), y(\tau)) = U_i(\tau)$, $(i = 1,\cdots,n)$, and the requirement that the canonical hyperbolic system (4.2) must be satisfied at each point of Υ. For, since $\dot{x} + \dot{y}^2 \neq 0$ along Υ, we may write the strip conditions

$$(4.10)\qquad \dot{u}_i = p_i\dot{x} + q_i\dot{y}, \quad (i = 1,\cdots,n),$$

as one of

$$(4.11)\quad q_i = \frac{1}{\dot{y}}(\dot{u}_i - p_i\dot{x}) \quad \text{or } p_i = \frac{1}{\dot{x}}(\dot{u}_i - q_i\dot{y}), \ (i = 1,\cdots,n).$$

Consider a particular point $P \in \Upsilon$ where $\dot{y} \neq 0$. Here we substitute $q_i = u_{i,y} = \frac{1}{\dot{y}}(\dot{u}_i - p_i\dot{x})$ into equations $B_i(P) = 0$, $(i=m+1,\cdots,n)$. These, together with the equations $A_i(P) = 0$, $(i = 1,\cdots, m<n)$,

form a linear algebraic system in the $p_i = u_{i,x}(P)$ with determinant $|a_{ik}| \neq 0$. Thus the p_i are uniquely determined at P, and, by (4.11), the q_i as well are uniquely determined at P. If $\dot{y} = 0$ at P, then $\dot{x} \neq 0$ there and a similar argument applies utilizing $p_i = \frac{1}{\dot{x}}(d_i - q_i\dot{y})$.

Thus we have, in effect, prescribed all three sets u_i, $u_{i,x}$, $u_{i,y}$, $(i = 1,\cdots,n)$, along Υ once the u_i are prescribed along Υ and the $u_{i,x}$ and the $u_{i,y}$ are merely required to satisfy the strip conditions (4.10) and the canonical hyperbolic system at (4.2) at each point of Υ.

Suppose we have a set of functions $\{u_1,\cdots, u_n\}$ as a solution of

(4.7) $u_{i,xy} = f_i(x,y; u_j; u_{j,x}, u_{j,y})$, $(i = 1,\cdots,n)$ in a neighborhood of the initial curve Υ and taking, with their first derivatives, precisely the above determined values at each point of Υ. Then by (4.5) and (4.6), the fact that these functions and their first derivatives satisfy the canonical hyperbolic system (4.2) at each point of Υ implies further that the system (4.2) corresponding to (4.7) is satisfied everywhere in the neighborhood in question.

With hypothesis 2) imposed, system (4.7) and the initial data on Υ satisfy the hypotheses of Theorem 5, while without hypothesis 2), system (4.7) and the initial data on Υ satisfy the hypotheses of Theorem 5a. But since we have shown above that each solution of (4.7) is a solution of the corresponding canonical

hyperbolic system (4.2), we have that Theorem 7 is a consequence of Theorem 5, while Theorem 7a is a consequence of Theorem 5a.

In these four theorems we are unable to prescribe the domain of definition of the functions

$$f_i(x,y;\ u_j;\ p_j,q_j),\ (i = 1,\cdots,n),$$

in such a way as to insure existence of a solution throughout $R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$. This is because the f_i are continuous for all p_j and q_j, $(j = 1,\cdots,n)$, but may turn out to be bounded only when these variables are restricted to finite domains. The following example demonstrates why the existence of solutions can be found only "in the small".

Example 3. Consider the characteristic initial value problem for the system

$$\begin{array}{lll} u_{1,xy} = u_{1,x}^2, & u_1(x,-1) = x, & u_1(0,y) = 0 \\ u_{2,xy} = 0, & u_2(x,-1) = 0, & u_2(0,y) = 0 \\ \vdots & \vdots & \\ u_{n,xy} = 0, & u_n(x,-1) = 0, & u_n(0,y) = 0. \end{array}$$

By quadratures, we obtain the solution $u_1(x,y) = \frac{-x}{y}$, while $u_2 = \cdots = u_n = 0$, quite obviously. The f_i corresponding to this problem possess derivatives of all orders for all values of all variables. However, $f_1 = u_{1,x}^2$ becomes unbounded as the argument $u_{1,x}$ increases indefinitely in absolute value. We note that, despite the specification of initial data everywhere along the

intersecting characteristics $x = 0$ and $y = -1$, the first function in the solution, namely u_1, has a discontinuity across the line $y = 0$. Hence this example typifies those cases for which solutions exist "in the small" only.

We note that Remark d) of Chapter III applies as well to hypothesis 5) of Theorems 7 and 7a. The statement is that

$$\Upsilon : \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases}$$

for $\tau \in [0,1]$ need only have $x(\tau)$ and $y(\tau) \in C'([0,1])$, monotone, and with $\dot{x}^2 + \dot{y}^2 \neq 0$ at each point of Υ. In fact, the argument in the proof above applies directly to this statement.

CHAPTER V.

The Cauchy Problem for $F(x,y;\,u;\,p,q;\,r,s,t) = 0$.

In this chapter we concern ourselves with the Cauchy problem for the general non-linear second order partial differential equation in the hyperbolic domain. Specifically, the problem is to determine an integral surface of the equation

$$(1.1)\qquad F(x,y;\,u;\,p,q;\,r,s,t) = 0$$

such that the hyperbolic condition

$$(1.3)\qquad F_s^{\,2} - 4\,F_r\,F_t > 0$$

is satisfied thereon; moreover, the integral surface must have a second order contact with a given second order strip which is nowhere a characteristic strip.

In his celebrated paper [10], H. LEWY successfully attacks this problem by reducing equation (1.1) to a system of first order partial differential equations for the unknowns $x,y;\,u;\,p,q;\,r,s,t$ as functions of the parameters λ and μ of the two families of characteristics on the integral surface in question. LEWY's existence proof for the system is based on a finite difference argument. However, the system is of canonical hyperbolic form and the theorem of M. CINQUINI-CIBRARIO, Theorem 7 of Chapter IV, is immediately applicable and insures existence and uniqueness of the solution in a sufficiently small neighborhood of the initial strip. Moreover, as demonstrated below, Theorem 7a may be used to effect an improvement on LEWY's work.

We present simultaneously LEWY's original theorem and our

improvement on it. LEWY's theorem is obtained by omitting the parentheses. Our theorem is obtained by replacing the underscored statements by the corresponding ones in the parentheses.

Theorem 8 (8a)

1) $S^2: \begin{cases} x = x(\tau) \\ y = y(\tau) \\ u = u(\tau) \\ p = p(\tau) \\ q = q(\tau) \\ r = r(\tau) \\ s = s(\tau) \\ t = t(\tau) \end{cases}$ for $\tau \in [0,1]$ is a nowhere characteristic second order strip,

i.e. $x,y;\ u;\ p,q;\ r,s,t(\tau) \in C'([0,1])$, and for each $\tau \in [0,1]$,

i) $\dot{x}^2 + \dot{y}^2 \neq 0$,

ii) $F_r\, \dot{y}^2 - F_s\, \dot{y}\, \dot{x} + F_t\, \dot{x}^2 \neq 0$,

iii) $F_s^{\,2} - 4\, F_r\, F_t > 0$,

iv) $F(x(\tau),\ y(\tau);\ u(\tau);\ p(\tau), q(\tau);\ r(\tau), s(\tau), t(\tau))$
$= 0$.

2) $\underline{F \in C'''}$ $(\underline{\in C''})$ in a certain neighborhood of S^2.

3) There exists one and only one (at least one) integral surface J: $u = u(x,y)$ of the equation $F(x,y;\ u;\ p,q;\ r,s,t) = 0$ such that $u(x,y) \in C'''$ in a sufficiently small neighborhood of the base curve $\Upsilon : \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases}$ for $\tau \in [0,1]$, and such that J: $u = u(x,y)$ has a second order contact with the strip S^2.

Proof

We first demonstrate that any solution of the above problem, together with its derivatives of the first and second orders, represents a solution of a particular canonical hyperbolic system under the same boundary conditions.

We assume that $F_r \neq 0$ and $F_t \neq 0$ in the domains considered in the following argument. This may be done without loss of generality. For, by Definition 1a, a characteristic base curve must satisfy

$$(1.5) \quad \begin{aligned} &1) \quad F_r \dot{y}^2 - F_s \dot{y} \dot{x} + F_t \dot{x}^2 = 0, \\ &2) \quad \dot{x}^2 + \dot{y}^2 \neq 0 . \end{aligned}$$

Suppose at a point of S^2 that $F_r = 0$. Then $\dot{x} = 0$ represents the vertical tangent taken by one of the characteristic base curves through the projection of this point onto the xy plane. Conversely, if one of the characteristic base curves through a point in the projection of S^2 has a vertical tangent, then $\dot{x} = 0$ there and, consequently, $F_r = 0$ at the corresponding point on S^2. Likewise, $F_t = 0$ if and only if $\dot{y} = 0$, in the sense above. Thus, by a suitable coordinate rotation in the xy plane, we may insure that $F_r \neq 0$ and $F_t \neq 0$ in a neighborhood of the point in question on S^2. Granting that this is a local property only and that the particular rotation performed may introduce values of $F_r = 0$ or $F_t = 0$ at some other sufficiently distant points on S^2, we observe that this local property is sufficient because our proof is ultimately based upon Theorems 4 and 4a of Chapter III. In those

theorems the integral equation statement of the problem made it plainly evident that the value of the integral at any point P depended only upon the portion of the initial curve cut off by the two characteristics intersecting at P. Consequently, we may consider the arguments below as applying in succession to small overlapping segments of S^2, with coordinate axes rotated suitably for each segment considered. (See also R. COURANT - D. HILBERT [17] p. 323 and p. 332.)

Let us assume that we have an integral surface J: u=u(x,y) satisfying the conditions of either Theorem 8 or Theorem 8a. Then by (1.5) we conclude that the related characteristic base curves are the two one-parameter families of curves determined by the equations

$$y_\lambda = \rho_1 x_\lambda , \tag{5.1}$$

$$y_\mu = \rho_2 x_\mu , \tag{5.2}$$

where

$$\rho_1 = \frac{F_s + \sqrt{F_s^2 - 4 F_r F_t}}{2 F_r} , \tag{5.3}$$

$$\rho_2 = \frac{F_s - \sqrt{F_s^2 - 4 F_r F_t}}{2 F_r} . \tag{5.4}$$

ρ_1 and ρ_2 are functions of the variables x,y; u; p,q; r,s,t and $\rho_1 \neq \rho_2$ in a neighborhood of S^2 by the hyperbolic condition (1.3).

Consider the coordinate transformation

$$\begin{aligned} x &= x(\lambda, \mu) \\ y &= y(\lambda, \mu) . \end{aligned} \tag{5.5}$$

The Jacobian of this transformation,

$$(5.6)\qquad y_\lambda x_\mu - y_\mu x_\lambda = (\rho_1 - \rho_2)\, x_\lambda x_\mu ,$$

does not vanish in a vicinity of the projection of S^2. This follows since $\rho_1 \neq \rho_2$; while $x_\lambda = 0$ would, by (5.1), imply $y_\lambda = 0$, contradicting the requirement $\dot{x}^2 + \dot{y}^2 \neq 0$, (similarly for x_μ). Hence the inverse transformation,

$$(5.7)\qquad \begin{cases} \lambda = \lambda\,(x,y) \\ \mu = \mu\,(x,y) \end{cases} ,$$

exists in a vicinity of the projection of S^2.

Along the characteristics on J: $u=u(x,y)$ certain additional equations must be satisfied. These are determined as follows:

Since $F \in C'''(\in C'')$ and $u \in C'''$, we obtain by differentiation

$$(5.8)\qquad \begin{cases} F_r r_x + F_s s_x + F_t t_x = -[F]_x \\ x_\lambda r_x + y_\lambda s_x = r_\lambda \\ \qquad x_\lambda s_x + y_\lambda t_x = s_\lambda , \end{cases}$$

where

$$(5.9)\qquad [F]_x = F_p r + F_q s + F_u p + F_x .$$

Similarly,

$$(5.10)\qquad \begin{cases} F_r r_y + F_s s_y + F_t t_y = -[F]_y \\ x_\lambda r_y + y_\lambda s_y = s_\lambda \\ \qquad x_\lambda s_y + y_\lambda t_y = t_\lambda , \end{cases}$$

where

(5.11) $$[F]_y = F_p s + F_q t + F_u q + F_y \,.$$

Since λ is the parameter for one family of characteristic curves and, consequently, is the path parameter along each of the curves of the other family, the determinant

(5.12) $$\begin{vmatrix} F_r & F_s & F_t \\ x_\lambda & y_\lambda & 0 \\ 0 & x_\lambda & y_\lambda \end{vmatrix} = F_r y_\lambda^2 - F_s y_\lambda x_\lambda \quad F_t x_\lambda^2 = 0.$$

Hence the quantities on the right-hand side in each of the systems (5.8) and (5.10) must be linearly dependent, i.e. in each system the augmented matrix of coefficients must be of rank less than three. Consequently,

(5.13) $$\begin{vmatrix} F_r & F_t & [F]_x \\ x_\lambda & 0 & -r_\lambda \\ 0 & y_\lambda & -s_\lambda \end{vmatrix} = F_r r_\lambda y_\lambda + F_t s_\lambda x_\lambda + [F]_x x_\lambda y_\lambda = 0.$$

Recalling the assumption made without loss,

$x_\lambda = \frac{1}{\rho_1} y_\lambda$ and $y_\lambda \not\equiv 0$, equation (5.13) reduces to

(5.14) $$F_r r_\lambda + \frac{1}{\rho_1} F_t s_\lambda + [F]_x x_\lambda = 0.$$

Likewise, from (5.10) we obtain the linear dependence of the right-hand terms in the form

(5.15) $$\rho_1 F_r s_\lambda + F_t t_\lambda + [F]_y y_\lambda = 0.$$

Along the curves of the other family of characteristics the following relations must be satisfied. These are obtained in a

fashion completely analogous to that used in obtaining (5.14) and (5.15):

$$F_r\, r_\mu + \frac{1}{\rho_2} F_t s_\mu + [F]_x\, x_\mu = 0 \tag{5.16}$$

$$\rho_2 F_r s_\mu + F_t t_\mu + [F]_y\, y_\mu = 0. \tag{5.17}$$

In addition, the strip conditions

$$\dot{u} = p\,\dot{x} + q\,\dot{y} \tag{1.8}$$

$$\begin{cases} \dot{p} = r\,\dot{x} + s\,\dot{y} \\ \dot{q} = s\,\dot{x} + t\,\dot{y} \end{cases} \tag{1.9}$$

must be satisfied along any curve lying on J: u=u(x,y). In particular, they must be satisfied along any characteristic on J.

From equations (5.1), (5.2), (5.14) through (5.17), (1.8) and (1.9) we obtain the following system of "characteristic equations" i.e. equations which must be satisfied along the characteristics on any integral surface J:

$$\left.\begin{aligned} \varphi_1 &= y_\lambda - \rho_1 x_\lambda = 0 \\ \varphi_2 &= F_r r_\lambda + \frac{1}{\rho_1} F_t s_\lambda + [F]_x\, x_\lambda = 0 \\ \varphi_3 &= \rho_1 F_r s_\lambda + F_t t_\lambda + [F]_y\, y_\lambda = 0 \\ \varphi_4 &= u_\lambda - p\, x_\lambda - q\, y_\lambda = 0 \\ \varphi_5 &= p_\lambda - r\, x_\lambda - s\, y_\lambda = 0 \\ \varphi_6 &= q_\lambda - s\, x_\lambda - t\, y_\lambda = 0 \\ \\ \psi_1 &= y_\mu - \rho_2 x_\mu = 0 \\ \psi_2 &= F_r r_\mu + \frac{1}{\rho_2} F_t s_\mu + [F]_x\, x_\mu = 0 \end{aligned}\right\} \text{System A} \tag{5.18}$$

(5.18)
(continued)

$$\left.\begin{aligned}
\psi_3 &= \rho_2 F_r s_\mu + F_t t_\mu + [F]_y y_\mu = 0\\
\psi_4 &= u_\mu - p x_\mu - q y_\mu = 0\\
\psi_5 &= p_\mu - r x_\mu - s y_\mu = 0\\
\psi_6 &= q_\mu - s x_\mu - t y_\mu = 0
\end{aligned}\right\} \text{System B}$$

We observe that System A of (5.18) is of canonical hyperbolic form in x,y; u; p,q; r,s,t as functions of λ and μ. Since for Theorem 8, $F \in C'''$, while for Theorem 8a, $F \in C''$, the coefficients of all equations in (5.18) are functions of class C'' for Theorem 8, and of class C' for Theorem 8a. Moreover, the determinant of the matrix of coefficients for System A, is, after interchange of rows and columns,

$$(5.19)\qquad \begin{vmatrix}
-\rho_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
-\rho_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
* & 0 & F_r & \frac{1}{\rho_1}F_t & 0 & 0 & 0 & 0\\
0 & * & 0 & \rho_1 F_r & F_t & 0 & 0 & 0\\
* & 0 & F_r & \frac{1}{\rho_2}F_t & 0 & 0 & 0 & 0\\
* & * & 0 & 0 & 0 & 1 & 0 & 0\\
* & * & 0 & 0 & 0 & 0 & 1 & 0\\
* & * & 0 & 0 & 0 & 0 & 0 & 1
\end{vmatrix}$$

$$= F_r F_t^2 \cdot \frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2},$$

where the coefficients designated only by asterisks, *, do not contribute to the value of the determinant. Since $F_r \neq 0$, $F_t \neq 0$ and $\rho_1 \neq \rho_2$ in a neighborhood of S^2, the determinant (5.19) does not vanish therein. Hence any solution J: u=u(x,y) of the problem of Theorem 8, together with its first and second derivatives,

satisfies the hypotheses for Theorem 7; because the requirement that $F \in C'''$ is certainly sufficient to insure that the first derivatives of the coefficients of System A be Lipschitzian with respect to variables x,y; u; p,q; r,s,t. Moreover, the requirement in Theorem 8a that $F \in C''$ insures that the coefficients of System A are of class C', as demanded by Theorem 7a.

In the $\lambda\mu$, or characteristic, plane, the initial base curve has the parametric form

$$\Upsilon: \begin{cases} \lambda = \lambda(x(\tau), y(\tau)) & \text{for } \tau \in [0,1], \\ \mu = \mu(x(\tau), y(\tau)) \end{cases}$$

and is nowhere parallel to either the λ or μ axes. Consequently, Υ may be expressed in the non-parametric form

$$\lambda = \varphi(\mu)$$

where $\varphi(\mu) \in C'$ and $\varphi'(\mu) \neq 0$. If we introduce $\lambda' = \lambda$ and $\mu' = -\varphi(\mu)$ as new characteristic parameters, we observe that equations (5.18) remain unaltered in form. Hence we may assume, without loss, that the initial base curve Υ has the representation

$$\lambda + \mu = 0 \tag{5.20}$$

in the $\lambda\mu$ plane.

We now demonstrate that any solution of System A satisfying the given Cauchy initial conditions is also a solution of the problem of Theorems 8 and 8a. This done, Theorems 8 and 8a are immediate consequences of Theorems 7 and 7a, respectively.

Following J. HADAMARD [11] p. 504, we show that for each set of functions satisfying System A and the initial conditions on

$\lambda + \mu = 0$, the System B is likewise satisfied. Note that in this part of the argument we cannot admit that p, q, r, s and t are derivatives of u. This is now a matter of proof.

Differentiating $F(x,y;\ u;\ p,q;\ r,s,t)$ by λ and observing equations (5.18), we obtain

$$(5.21)\qquad \frac{dF}{d\lambda} = \varphi_2 + \varphi_3 + F_u\varphi_4 + F_p\varphi_5 + F_q\varphi_6.$$

Hence $\frac{dF}{d\lambda} = 0$ for each set of functions satisfying System A. However, by hypothesis, $F = 0$ along $\lambda + \mu = 0$. Thus $F \equiv 0$ throughout that region where the set of functions satisfying System A is defined. This in turn implies that

(5.22) $\frac{dF}{d\mu} = \psi_2 + \psi_3 + F_u\psi_4 + F_p\psi_5 + F_q\psi_6 = 0$ throughout the same region. By hypothesis, $\psi_2 = 0$ in this region, hence

$$(5.23)\qquad \psi_3 = -F_u\psi_4 - F_p\psi_5 - F_q\psi_6$$

therein.

Since $\rho_1\rho_2 = \frac{F_t}{F_r}$, we obtain from (5.18) by simple algebraic operations

$$(5.24)\qquad \frac{\rho_1 y_\mu}{F_t}\varphi_2 = r_\lambda x_\mu + s_\lambda y_\mu + H,$$

$$(5.25)\qquad \frac{\rho_2 y_\lambda}{F_t}\psi_2 = r_\mu x_\lambda + s_\mu y_\lambda + H,$$

where

$$(5.26)\qquad H = \frac{y_\lambda y_\mu}{F_t}[F]_x = \frac{x_\lambda x_\mu}{F_r}[F]_x\ ;$$

$$(5.27)\qquad \frac{y_\mu}{F_t}\varphi_3 = s_\lambda x_\mu + t_\lambda y_\mu + E,$$

$$(5.28)\qquad \frac{y_\lambda}{F_t}\,\psi_3 = s_\mu x_\lambda + t_\mu y_\lambda + K,$$

where

$$(5.29)\qquad K = \frac{y_\lambda y_\mu}{F_t}\,[F]_y = \frac{x_\lambda x_\mu}{F_r}\,[F]_y\ .$$

By Theorem 7 or Theorem 7a, the functions of the set satisfying System A and the Cauchy initial data are continuously differentiable and possess continuous mixed second derivatives. Thus we may perform the differentiations in the following relations:

$$(5.30)\qquad \begin{aligned}\psi_{4,\lambda} - \varphi_{4,\mu} &= p_\lambda x_\mu + q_\lambda y_\mu - p_\mu x_\lambda - q_\mu y_\lambda\\ &= \varphi_5 x_\mu - \varphi_6 y_\mu - \psi_5 x_\lambda - \psi_6 y_\lambda\ ;\end{aligned}$$

$$(5.31)\qquad \begin{aligned}\psi_{5,\lambda} - \varphi_{5,\mu} &= r_\lambda x_\mu + s_\lambda y_\mu - r_\mu x_\lambda - s_\mu y_\lambda\\ &= \frac{\rho_1 y_\mu}{F_t}\,\varphi_2 - \frac{\rho_2 y_\lambda}{F_t}\,\psi_2,\end{aligned}$$

by (5.24) and (5.25) above;

$$(5.32)\qquad \begin{aligned}\psi_{6,\lambda} - \varphi_{6,\mu} &= s_\mu x_\lambda + t_\mu y_\lambda - s_\lambda x_\mu - t_\lambda y_\mu\\ &= \frac{y_\lambda}{F_t}\,\psi_3 - \frac{y_\mu}{F_t}\,\varphi_3\ ,\end{aligned}$$

by (5.27) and (5.28) above. But System A is satisfied, hence (5.30), (5.31) and (5.32), by virtue of (5.23), reduce to

$$(5.33)\qquad \begin{cases}\psi_{4,\lambda} = -\,\psi_5 x_\lambda - \psi_6 y_\lambda\\ \psi_{5,\lambda} = 0\\ \psi_{6,\lambda} = \dfrac{-y_\lambda}{F_t}\,(F_u\,\psi_4 + F_p\,\psi_5 + F_q\,\psi_6).\end{cases}$$

In (5.33) all functions are known except ψ_4, ψ_5, ψ_6 and their derivatives with respect to λ. Moreover, along $\lambda = -\mu$ System B is satisfied, i.e. $\psi_4 = \psi_5 = \psi_6 = 0$ for $\lambda = -\mu$. For fixed μ we may consider (5.33) as a homogeneous system of linear first order ordinary differential equations under homogeneous onepoint boundary conditions. This system has the unique solution

$$\psi_4 = \psi_5 = \psi_6 = 0$$

throughout the region of definition of the set of functions satisfying System A. By (5.23), $\psi_3 = 0$ also, and the System B is shown to be dependent upon the System A in the sense above.

From the functions $x = x(\lambda, \mu)$, $y = y(\lambda, \mu)$ of the set satisfying System A, we may form the inverse functions $\lambda = \lambda(x, y)$, $\mu = \mu(x, y)$, since the Jacobian

(5.6) $$y_\lambda x_\mu - y_\mu x_\lambda = (\rho_1 - \rho_2)\, x_\lambda x_\mu$$

does not vanish. Hence we may express the function $u = u(\lambda, \mu)$ as a function of the independent variables x and y.

We now need to show only that

(5.34) $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$ and $t = u_{yy}$

throughout the above region to complete the proof.

Now
$$\varphi_4 = u_\lambda - px_\lambda - qy_\lambda = 0$$
$$\psi_4 = u_\mu - px_\mu - qy_\mu = 0,$$

while the determinant of this linear system is the Jacobian (5.6) and hence does not vanish. Thus there exists a unique solution.

90. W. M. WHYBURN, "Over and under functions as related to differential equations," American Mathematical Monthly, vol. 47 (1940), pp. 1-10.

But $p = u_x$, $q = u_y$ obviously satisfies and hence represents the unique solution.

Similarly,

$$\varphi_5 = u_{x,\lambda} - rx_\lambda - sy_\lambda = 0$$
$$\psi_5 = u_{x,\mu} - rx_\mu - sy_\mu = 0,$$

hence $r = u_{xx}$ and $s = u_{xy}$;

$$\varphi_6 = u_{y,\lambda} - sx_\lambda - ty_\lambda = 0$$
$$\psi_6 = u_{y,\mu} - sx_\mu - ty_\mu = 0,$$

hence $t = u_{yy}$ and $u_{yx} = u_{xy} = s$. The proof is now complete.

CHAPTER VI

The Characteristic Initial Value Problem for $F(x,y;u;p,q;\ r,s,t) = 0$.

The whole idea of a characteristic initial value problem for the equation

$$(1.1) \qquad F(x,y;\ u;\ p,q;\ r,s,t) = 0$$

appears paradoxical at first glance. In the Cauchy problem the prescribed initial data was sufficient to determine whether or not the projection of the initial curve was characteristic. In this problem, however, we merely prescribe two intersecting space curves through which an integral surface of the equation (1.1) must pass. Since the characteristics are, in general, dependent on the integral surface in question, it would appear impossible to determine, a priori, whether or not the prescribed initial curves have characteristic projections.

That such is not the case is demonstrated by M. CINQUINI-CIBRARIO [13]. In this paper she treats the characteristic initial value problem as a special case of the more general Goursat problem, i.e. where two arbitrary intersecting space curves are prescribed through which an integral surface of (1.1) must pass. Commencing on p. 220, she gives the necessary and sufficient conditions that these curves be characteristic to any integral surface passing through them. We call curves satisfying these conditions "intrinsically characteristic" curves.

In this chapter we examine her development, for the particular case of the characteristic initial value problem, up to the point where a modified form of the system of characteristic equations (5.18) and the above necessary and sufficient conditions are obtained. There are two important differences between her development and that of H. LEWY given in the preceding chapter. First, she transforms the initial curves into the coordinate axes. Since these curves are characteristic, this implies immediately that $F_r = 0$ and $F_t = 0$ at the origin. Thus many of the divisions performed in Chapter V are now invalidated. Second, she is able to solve (1.1) explicitly for s, obtaining

$$s = f(x,y;\ u;\ p,q;\ r,t)$$

and thus to reduce the number of equations in the system of characteristic equations by two.

We do not follow the remainder of her existence proof, in which she reduces the system of characteristic equations to an integral equation form and then applies successive approximations to obtain the existence of a unique solution to the general Goursat problem. Instead we deal directly with the special case of the characteristic initial value problem by a method analogous to that of Chapter V. Such an approach is indicated by M. CINQUINI-CIBRARIO, herself, [12] p.180, footnote 8. She states, in effect, that the following Theorem 9 can be shown to be a consequence of Theorem 6, Chapter IV. We present this proof in detail and, in addition, we extend it to apply to the derivation of Theorem 9a as a consequence of Theorem 6a. The improvement obtained corresponds to that of

Chapter V for the Cauchy problem. Namely, the requirement that $F \in C'''$ is reduced to require merely that $F \in C''$ while the conclusion is altered to read "at least one solution" instead of "one and only one solution".

Theorem 9

1)
$$\Gamma_1: \begin{cases} \gamma_1: \begin{cases} x_1 - \xi \leq x \leq x_1 + \xi \\ y = f_1(x) \end{cases} & , \; f_1(x) \in C''([x_1 - \xi, x_1 + \xi]) \\ \qquad\quad u = F_1(x) & \quad F_1(x) \in C''([x_1 - \xi, x_1 + \xi]). \end{cases}$$

$$\Gamma_2: \begin{cases} \gamma_2: \begin{cases} x = f_2(y) \\ y_1 - \eta \leq y \leq y_1 + \eta \end{cases} & , \quad f_2(y) \in C''([y_1 - \eta, y_1 + \eta]) \\ \qquad\quad u = F_2(y) & \qquad F_2(y) \in C''([y_1 - \eta, y_1 + \eta]) \end{cases}$$

The point (x_1, y_1) is the only point of intersection of γ_1 and γ_2 and it is interior to both curves. Moreover, $F_1(x_1) = F_2(y_1)$ and $f_1'(x_1) f_2'(y_1) \neq 1$. (i.e. γ_1 and γ_2 do not have a common tangent at the point (x_1, y_1).)

2) Γ_1 and Γ_2 are "intrinsically characteristic" in a neighborhood of their point of intersection, i.e. they meet the necessary and sufficient conditions, given below, that they be characteristic to any integral surface of

$$(1.1) \qquad F(x, y; u; p, q; r, s, t) = 0$$

passing through them. As we shall see below, this hypothesis, together with hypothesis 1), tacitly implies that at the intersection point (x_1, y_1, u_1) of Γ_1 and Γ_2 the values p_1, q_1, r_1, s_1,

t_1), the hyperbolic condition

$$F_{s_1}^{\,2} - 4\, F_{r_1} F_{t_1} > 0,$$

is satisfied, (notation: $F_{s_1} = F_s(x_1, y_1;\ u_1;\ p_1, q_1; r_1, s_1, t_1)$, etc.)

3) $F \in C'''$ in a neighborhood of the point

$$(x_1, y_1;\ u_1;\ p_1, q_1;\ r_1, s_1, t_1).$$

⟹ 4) There exists one and only one integral surface $J: u = u(x,y)$ of $F(x,y;\ u; p,q;\ r,s,t) = 0$, defined and of class C''' in a sufficiently small neighborhood of the point (x_1, y_1) and passing through subarcs of Γ_1 and Γ_2 intersecting at the point (x_1, y_1, u_1).

Theorem 9a

1)

2)

3)' $F \in C''$ in a neighborhood of the point

$$(x_1, y_1;\ u_1;\ p_1, q_1; r_1, s_1, t_1).$$

⟹ 4)' There exists at least one integral surface etc. (as in Theorem 9).

Proof of Theorems 9 and 9a

We first perform the coordinate transformation

$$(6.1) \qquad \begin{cases} \bar{x} = x - f_2(y) \\ \bar{y} = y - f_1(x) \end{cases}$$

taking γ_1 into the $\bar{x}$ axis, γ_2 into the $\bar{y}$ axis and the point (x_1, y_1) into the origin. This transformation is univalent in a

neighborhood of (x_1,y_1) since the Jacobian

$$(6.2) \qquad 1 - f_1'(x_1)f_2'(y_1) \neq 0$$

by hypothesis 1). Geometrically, this means that γ_1 and γ_2 do not have a common tangent at their point of intersection.

Without loss, we may assume homogeneous initial conditions. For, suppose we have an integral surface J: $u=u(x,y)$ of equation (1.1) passing through the curves Γ_1 and Γ_2. Then by the above transformation, considering (6.2),

$$(6.3) \qquad u(x,y) = \bar{u}(\bar{x}(x,y),\ \bar{y}(x,y)),$$

and hence for any such integral surface

$$(6.4) \quad \begin{cases} F_1(x) = u(x,f_1(x)) = u(\bar{x}(x,f_1(x)),\ 0), \\ F_2(y) = u(f_2(y),y) = \bar{u}(\ \ 0,\bar{y}(f_2(y),y)). \end{cases}$$

Letting

$$(6.5) \quad w(\bar{x},\bar{y}) = \bar{u}(\bar{x},\bar{y}) - \bar{u}(\bar{x},0) - \bar{u}(0,\bar{y}) + \bar{u}(0,0),$$

and since, by hypothesis 1), f_1, f_2, F_1 and $F_2 \in C''$, we obtain

$$(6.6) \quad \begin{cases} w(\bar{x},0) = w_{\bar{x}}(\bar{x},0) = w_{\bar{x}\bar{x}}(\bar{x},0) = 0, \\ w(0,\bar{y}) = w_{\bar{y}}(0,\bar{y}) = w_{\bar{y}\bar{y}}(0,\bar{y}) = 0. \end{cases}$$

Thus we may reduce the problem to that of finding a function $w = w(\bar{x},\bar{y})$ which vanishes on the coordinate axes in a vicinity of the origin and satisfies there the transformed form of equation (1.1),

$$(6.7)\quad F(\bar{x},\bar{y};[w+g];\ [w+g]_{,\bar{x}},\ [w+g]_{,\bar{y}};\ [w+g]_{,\bar{x}\bar{x}},$$
$$[w+g]_{,\bar{x}\bar{y}},\ [w+g]_{,\overline{yy}})$$

where

$$(6.8)\qquad g(\bar{x},\bar{y}) = \bar{u}(\bar{x},0) + \bar{u}(0,\bar{y}) - \bar{u}(0,0).$$

The function g is known from the prescribed initial data.

For simplicity, we return to our original notation and state the problem in this way:

To determine the function $u = u(x,y)$ satisfying equation (1.1) and the initial conditions

$$u(x,0) = u(0,y) = 0,$$

where, in the notation above,

$$u_0 = p_0 = q_0 = r_0 = t_0 = 0$$

and

$$(6.9)\qquad F(0,0;\ 0;\ 0,0;\ 0,s_0 0) = 0.$$

By hypothesis 2), there exists a unique value s_0 satisfying (6.9).

The characteristic base curves and, a fortiori, the hyperbolic condition are invariant under the transformation (6.1). (See R. COURANT - D. HILBERT [17] p. 304.) Moreover, the substitution $w = \bar{u} - g$ also preserves the invariance of the equation for the characteristic base curves and the hyperbolic condition as is easily seen by differentiation of (6.7). Hence, by hypothesis 2), we have the hyperbolic condition

(6.10) $$F_{s_0}^2 - 4 F_{r_0} F_{t_0} > 0,$$

while the equation for the characteristic base curve directions at the origin is

(6.11) $$F_{r_0} dy^2 - F_{s_0} dxdy + F_{t_0} dx^2 = 0.$$

Hypothesis 2) implies that the coordinate axes must be characteristic base curves. By (6.11) and (6.10) this in turn implies that $F_{r_0} = F_{t_0} = 0$, and hence that $F_{s_0} \neq 0$. But now the Implicit Function Theorem tells us that in the neighborhood of the point $(0,0;\ 0;\ 0,0;\ 0,\ s_0,\ 0)$ equation (1.1) can be solved explicitly in the form

(6.12) $$s = f(x,y;\ u;\ p,q;\ r,t).$$

Under hypothesis 3) or 3)', the function $f \in C'''$ or C'', respectively, in a neighborhood of this point. Moreover,

(6.13) $$f_{r_0} = f_{t_0} = 0 \quad \text{and} \quad s_0 = f_0$$

while the hyperbolic condition becomes at the origin

(6.14) $$1 - 4 f_{r_0} f_{t_0} = 1 > 0$$

and the equation for the characteristic base curves becomes

(6.15) $$f_r\, dy^2 + dxdy + f_t dx^2 = 0.$$

Let us assume that we have a particular integral surface $J:\ u = u(x,y)$ passing through the coordinate axes in a neighborhood of the origin, with $u(x,y) \in C'''$ in this neighborhood.. We define

$$(6.16) \qquad \delta = \sqrt{1-4\, f_r f_t}, \quad \rho = \frac{-2f_t}{1+\delta}, \; \sigma = \frac{-2f_r}{1+\delta},$$

δ, ρ and σ being of class C'' by hypothesis 3), or of class C' by hypothesis 3)', in the variables x,y; u; p,q; r,t in a neighborhood of the point (0,0; 0; 0,0; 0,0). The two one-parameter families of characteristic base curves corresponding to J are thus represented by the equations

$$(6.17) \qquad y_\lambda = \rho x_\lambda$$

$$(6.18) \qquad x_\mu = \sigma y_\mu .$$

Note that $\delta_o = 1$, hence $\delta > 0$ in a neighborhood of the origin, while $\rho_o = \sigma_o = 0$.

As in Chapter V, to obtain the system of characteristic equations, we transform to the characteristic base curves as coordinates and consider what relations must be satisfied along these coordinates for any given integral surface J. In particular, we specialize the transformation

$$(6.19) \qquad \begin{cases} x = x(\lambda, \mu) \\ y = y(\lambda, \mu) \end{cases}$$

by stipulating that a line λ = constant shall have x-intercept $(\lambda, 0)$ and a line μ = constant shall have y-intercept $(0, \mu)$, with $\lambda = \mu = 0$ at the origin. The Jacobian of this transformation, evaluated at the origin, has the value

$$(6.20) \quad x_{\lambda_o} y_{\mu_o} - y_{\lambda_o} x_{\mu_o} = x_{\lambda_o} y_{\mu_o}(1-\rho_o\sigma_o) = x_{\lambda_o} y_{\mu_o} \neq 0,$$

since if $x_{\lambda_o} = 0$, then $y_{\lambda_o} = 0$ by (6.17), contradicting the requirement that $\dot{x}^2 + \dot{y}^2 \neq 0$ along any characteristic curve.

Similarly, if $y_{\mu_0} = 0$, then $x_{\mu_0} = 0$ by (6.18) and the contradiction is again obtained.

Paralleling our development in Chapter V, we see that certain determinants must vanish at each point of the integral surface J, yielding equations which must be satisfied along the characteristics on J. We have

$$(6.21)\quad \begin{vmatrix} f_r & -[f]_x & f_t \\ x_\lambda & r_\lambda & 0 \\ 0 & s_\lambda & y_\lambda \end{vmatrix} = f_r r_\lambda y_\lambda + f_t s_\lambda x_\lambda + [f]_x x_\lambda y_\lambda = 0$$

where

$$(6.22)\quad [f]_x = f_p r + f_q s + f_u p + f_x .$$

also

$$(6.23)\quad \begin{vmatrix} f_r & -[f]_y & f_t \\ x_\lambda & s_\lambda & 0 \\ 0 & t_\lambda & y_\lambda \end{vmatrix} = f_r s_\lambda y_\lambda + f_t t_\lambda x_\lambda + [f]_y x_\lambda y_\lambda = 0$$

where

$$(6.24)\quad [f]_y = f_p s + f_q t + f_u q + f_y .$$

Eliminating s_λ between (6.21) and (6.23), we obtain

$$(6.25)\quad f_r^{\,2} r_\lambda y_\lambda^{\,2} - f_t^{\,2} t_\lambda x_\lambda^{\,2} + [f]_x f_r x_\lambda y_\lambda^{\,2} - [f]_y f_t x_\lambda^{\,2} y_\lambda = 0.$$

By virtue of definitions (6.16) and equation (5.17), we may write (6.25) as

$$(6.26)\quad f_t^{\,2} x_\lambda^{\,2} \cdot H(\lambda, \mu) = 0$$

where

$$(6.27) \quad H(\lambda,\mu) = r_\lambda \sigma^2 - t_\lambda + \frac{2}{1+\delta}\left\{[f]_y - \sigma[f]_x\right\} x_\lambda \, .$$

But, as shown above, $x_\lambda \neq 0$ along any of the characteristic base curves of J of the corresponding family, hence (6.26) reduces to

$$(6.28) \qquad f_t^2 \cdot H(\lambda,\mu) = 0.$$

Where $f_t \neq 0$ we have immediately that $H(\lambda,\mu) = 0$. Suppose at a particular point of J that $f_t = 0$. Then by (6.16) and (6.17), we have there that

$$(6.29) \quad \rho = 0, \quad \delta = 1, \quad \sigma = -f_r \quad \text{and } y_\lambda = 0.$$

Thus, at this point, by (6.24),

$$(6.30) \quad t_\lambda = s_y x_\lambda = (f_r r_y + [f]_y) x_\lambda ;$$

while by (5.22),

$$(6.31) \quad r_\lambda \sigma^2 = f_r^2 r_x x_\lambda = f_r^2 (s_\lambda - [f]_x x_\lambda).$$

Substituting (6.30) and (6.31) into (6.27), we obtain that where $f_t = 0$ on J, $H(\lambda,\mu) = 0$. Hence by (6.28), $H(\lambda,\mu) = 0$ everywhere on J and represents a relation which must be satisfied along each characteristic of the corresponding family on J.

For the other family of characteristics on J, we have determinants corresponding to (6.21) and (6.22) which vanish at each point of J. Eliminating s_μ between these and arguing in a fashion analogous to that above, we arrive at the following rela-

tion which must be satisfied along each characteristic of this family on J:

$$(6.32) \quad H(\lambda,\mu) = \rho^2 t_\mu - r_\mu + \frac{2}{1+\delta}\left\{[f]_x - \rho[f]_y\right\} y_\mu = 0.$$

We are now in a position to prescribe the necessary and sufficient conditions that the coordinate axes be characteristics for any integral surface of

$$(6.12) \qquad s = f(x,y;\, u;\, p,q;\, r,t)$$

passing through them.

Suppose that, in a neighborhood of the origin, the coordinate axes are characteristic to some integral surface J: $u=u(x,y)$ of (6.12) passing through them. Then in terms of the characteristic base curves to J as coordinates, defined by the coordinate transformation (6.19), we have for $\mu = 0$:

$$x = \lambda,\ y = 0,\ u = p = r = 0,\ q = Q(\lambda),\ t = T(\lambda),$$

where, from (6.12),

$$(6.33) \quad Q'(\lambda) = f(\lambda,0;\, 0;\, 0,\, Q(\lambda);\, 0,\, T(\lambda)),$$

while, from $H(\lambda,\mu) = 0$, since $\rho = f_t = 0$, $\delta = 1$ and $\sigma = -f_r$,

$$(6.34) \quad T'(\lambda) = \left\{[f]_y + f_r[f]_x\right\}(\lambda,0;\, 0;\, 0,\, Q(\lambda);\, 0, T(\lambda)).$$

Moreover,

$$(6.35) \qquad Q(0) = T(0) = 0.$$

Equations (6.33) and (6.34) represent a system of first order ordinary differential equations under one point boundary conditions (6.35). The right hand sides of the equations of this system are of class C'' under hypothesis 3), or of class C' under hypothesis 3)', in the variables λ, Q and T. Hence, in either case, the functions Q and T are uniquely determined in a neighborhood of $\lambda = 0$. If the x axis is characteristic, these functions must also satisfy

$$(6.36)\qquad f_t(\lambda, 0;\ 0;\ 0,\ Q(\lambda);\ 0,\ T(\lambda)) = 0.$$

Similarly, for $\lambda = 0$:

$x = 0,\ y = \mu,\ u = q = t = 0,\ p = P(\mu),\ r = R(\mu),$

where, from (6.12),

$$(6.37)\qquad P'(\mu) = f(0, \mu;\ 0;\ P(\mu), 0;\ R(\mu), 0),$$

while, from $X(\lambda, \mu) = 0$, since $\sigma = f_r = 0$, $\delta = 1$ and $\rho = - f_t$,

$$(6.38)\quad R'(\mu) = \left\{ [f]_x + f_t [f]_y \right\} (0, \mu;\ 0;\ P(\mu), 0;\ R(\mu), 0).$$

Moreover,

$$(6.39)\qquad P(0) = R(0) = 0.$$

Hence, if the y axis is characteristic, the functions P and R, uniquely determined by (6.37), (6.38), and (6.39), must also satisfy

$$(6.40)\qquad f_r(0, \mu;\ 0;\ P(\mu), 0;\ R(\mu), 0) = 0.$$

To recapitulate, the necessary condition that the x axis be a characteristic of some integral surface is that the functions Q and T determined from the system (6.33) and (6.34), under boundary conditions (6.35), shall satisfy (6.36) for each λ in a neighborhood of $\lambda = 0$. The necessary condition that the y axis be a characteristic of some integral surface is that the functions P and R determined from the system (6.37) and (6.38, under boundary conditions (6.39), shall satisfy (6.40) for each μ in a neighborhood of $\mu = 0$.

We now show that these conditions are also sufficient, i.e. given in the vicinity of the origin, an integral surface J: $u = u(x,y)$ of (6.12) passing through the coordinate axes, with

(6.41) $P_1(y) = u_x(o,y)$, $R_1(y) = u_{xx}(0,y)$, $Q_1(x) = u_y(x,0)$,
and $T_1(x) = u_{yy}(x,0)$,

we show that the requirement

(6.40)' $f_r(0,y;\ 0;\ P_1(y),0;\ R_1(y),0) = 0$

is sufficient that the y axis be a characteristic on J.

The argument needed to show that the requirement

(6.36)' $f_t(x,0;\ 0;\ 0,Q_1(x);\ 0,\ T_1(x)) = 0$

is sufficient in order that the x axis be a characteristic on J is analogous to the following and will not be given here.

We need show only that under requirement (6.40)', $P_1(y) = P(y)$ and $R_1(y) = R(y)$, where $P(y)$ and $R(y)$ are those functions obtained

previously under the assumption that the y-axis was "intrinsically characteristic".

Now $P_1(0) = R_1(0) = 0$ since $u(x,0) = 0$. Moreover, since u satisfies

$$s = f(x,y;\ u;\ p,q;\ r,t), \tag{6.19}$$

for $x = 0$,

$$P_1'(y) = f(0,y;\ 0;\ P_1(y),\ 0;\ R_1(y),0). \tag*{(6.37)'}$$

Now, recalling that $u \in C'''$,

$$s_x = f_r\, r_x + f_t\, t_x + [f]_x, \tag{6.42}$$

$$s_y = f_r\, r_y + f_t\, t_y + [f]_y. \tag{6.43}$$

Since $u(0,y) = 0$, we obtain $t_y(0,y) = 0$. Writing $r_x(0,y) = w(y)$ and substituting (6.43) into (6.42) with $x = 0$, we obtain

$$\begin{aligned} s_x(0,y) &= r_y(0,y) \\ &= f_r\, w(y) + f_t\, f_r\, r_y + [f]_x + f_t\, [f]_y \end{aligned} \tag{6.44}$$

But, $u(0,y) = u_y(0,y) = u_{yy}(0,y) = 0$, hence by (6.44),

$$R_1'(y) = \left[\frac{1}{1-f_r f_t}\left\{[f]_x + f_t[f]_y + f_r w(y)\right\}\right](0,y;\ 0;\ P_1(y),0;\ R_1(y),0). \tag*{(6.38)'}$$

Now equation (6.37)' is precisely the same as (6.37), while requirement (6.40)' is sufficient to reduce (6.38)' to (6.38). But this implies that $P_1(y) = P(y)$ and $R_1(y) = R(y)$ since the solution of the system of ordinary differential equations in question is unique.

In the foregoing arguments we have developed a procedure for determining whether or not the initial curves are "intrinsically characteristic". By transformation (6.1) and substitution (6.5), we reduce the initial curves Γ_1 and Γ_2 to the coordinate axes. If now a_0 can be uniquely determined from (6.9) we may verify the hyperbolic condition and obtain the characteristic directions at the origin. If these directions coincide with the coordinate axes, then equation (1.1) can be solved explicitly for (6.12). From this, the system (6.37) and (6.38) under boundary condition (6.39) can, in principle at least, be solved for functions P and R. Finally if P and R satisfy (6.40) then the y axis is characteristic to any integral surface of the problem, i.e. "intrinsically characteristic". Likewise, from the system (6.33) and (6.34) under boundary condition (6.35), the functions Q and T can be determined. If these satisfy (6.36) then the x axis is "intrinsically characteristic". Note that P, R, Q and T are evidently of class C'.

Having given hypothesis 2) a precise meaning along with a procedure for determining whether or not it is verified for a given problem, we continue with the proof under the assumption that hypothesis 2) is verified.

From equations (6.17), (6.18), (6.27), (6.32) and the strip conditions we obtain the following system of characteristic equations, which must be satisfied along the characteristics on any integral surface J:

$$(6.45)\qquad \left.\begin{array}{l}
\varphi_1 = y_\lambda - \rho x_\lambda = 0 \\
\varphi_2 = r_\lambda \sigma^2 - t_\lambda + \frac{2}{1+\delta}\left\{[f]_y - \sigma [f]_x\right\} x_\lambda = 0 \\
\varphi_3 = u_\lambda - px_\lambda - qy_\lambda = 0 \\
\varphi_4 = p_\lambda - rx_\lambda - fy_\lambda = 0 \\
\varphi_5 = q_\lambda - fx_\lambda - ty_\lambda = 0 \\
\\
\psi_1 = x_\mu - \sigma y_\mu = 0 \\
\psi_2 = r_\mu - \rho^2 t_\mu - \frac{2}{1+\delta}\left\{[f]_x - \rho [f]_y\right\} y_\mu = 0
\end{array}\right\}\text{System A}$$

$$\left.\begin{array}{l}
\psi_3 = u_\mu - px_\mu - qy_\mu = 0 \\
\psi_4 = p_\mu - rx_\mu - fy_\mu = 0 \\
\psi_5 = q_\mu - fx_\mu - ty_\mu = 0
\end{array}\right\}\text{System B}$$

We observe that System A of (6.45) is of canonical hyperbolic form in x,y; u; p,q; r,t as functions of λ and μ. Since for Theorem 9, $F \in C'''$, while for Theorem 9a, $F \in C''$, the coefficients of all equations in (6.45) are functions of class C'' for Theorem 9, and of class C' for Theorem 9a. Moreover, the matrix of coefficients for System A is, after interchange of rows and columns,

$$(6.46)\qquad \begin{vmatrix}
-\rho & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -\sigma & 0 & 0 & 0 & 0 & 0 \\
* & 0 & \sigma^2 & -1 & 0 & 0 & 0 \\
0 & * & 1 & -\rho^2 & 0 & 0 & 0 \\
* & * & 0 & 0 & 1 & 0 & 0 \\
* & * & 0 & 0 & 0 & 1 & 0 \\
* & * & 0 & 0 & 0 & 0 & 1
\end{vmatrix}$$

$$= (1 - \rho\sigma)(\sigma^2\rho^2 - 1) = \frac{-2\delta^2}{(1+\delta)^3}$$

where the coefficients designated only by asterisks, *, do not contribute to the value of the determinant. But $\delta > 0$ everywhere on J in a neighborhood of the origin, hence the determinant (6.46) does not vanish thereon.

As to the initial conditions, we have, by hypothesis 1) of Theorems 9 and 9a for $\mu = 0$,

$$x = \lambda, \; y = 0, \quad u = p = r = 0, \; q = Q(\lambda), \; t = T(\lambda),$$

and for $\lambda = 0$,

$$x = 0, \; y = \mu, \; u = q = t = 0, \quad p = P(\mu), \; r = R(\mu)$$

where Q, T and P,R are determined from their respective systems and are of class C'. Moreover, for $\mu = 0$, by (6.36), $f_t = 0$. Hence $\rho = 0$, $\delta = 1$, and $\sigma = -f_r$. This together with $y_\lambda = r_\lambda = u_\lambda = p_\lambda = 0$ and equation (6.34) prove that

$$(6.47) \quad \varphi_1(\lambda,0) = \varphi_2(\lambda,0) = \varphi_3(\lambda,0) = \varphi_4(\lambda,0) = \varphi_5(\lambda,0) = 0$$

for all λ in a neighborhood of $\lambda = 0$. Similarly, for $\lambda = 0$, by (6.40), $f_r = 0$. Hence $\sigma = 0$, $\delta = 1$ and $\rho = -f_t$. This together with $x_\mu = t_\mu = u_\mu = q_\mu = 0$ and equation (6.38) prove that

$$(6.48) \quad \psi_1(0,\mu) = \psi_2(0,\mu) = \psi_3(0,\mu) = \psi_4(0,\mu) = \psi_5(0,\mu) = 0$$

for all μ in a neighborhood of $\mu = 0$. Thus the initial condition requirements of hypothesis 3) of Theorems 6 and 6a are satisfied.

Since the coefficients in (6.45) are of class C'' for Theorem 9, hypotheses 1) and 2) of Theorem 6 are satisfied. Also, since the coefficients in (6.45) are of class C' for Theorem 9a, the

common hypothesis 1) of Theorems 6 and 6a is satisfied, but hypothesis 2) of Theorem 6, a hypothesis which does not appear in Theorem 6a, is not satisfied. Thus if we now show that any solution of the canonical hyperbolic system, System A of (6.45), with the given characteristic initial conditions is also a solution of the corresponding problem for the equation

$$(6.12) \qquad s = f(x,y;\, u;\, p,q;\, r,t)$$

with the same initial conditions, then Theorem 9 is an immediate consequence of Theorem 6 and Theorem 9a is an immediate consequence of Theorem 6a.

As in the Cauchy problem of Chapter 5, we show that for each solution of System A under the given characteristic initial conditions that System B is likewise satisfied. Note that here we cannot assume that p,q,r and t are derivatives of u; this is a matter of proof. Recalling from Theorems 6 and 6a that the functions of the solution of System A, x,y,u,p,q,r,t are of class C' and that $f \in C'''$ under hypothesis 3) of Theorem 9, or $f \in C''$ under hypothesis 3)' of Theorem 9a, we obtain by differentiation and consideration of (6.45) that

$$(6.49) \quad \begin{aligned} \psi_{3,\lambda} - \varphi_{3,\mu} &= p_\mu x_\lambda + q_\mu y_\lambda - p_\lambda x_\mu - q_\lambda y_\mu \\ &= \psi_4 x_\lambda + \psi_5 y_\lambda - \varphi_4 x_\mu - \varphi_5 y_\mu \,. \end{aligned}$$

Moreover, since $\varphi_3 = \varphi_4 = \varphi_5 = 0$,

$$(6.50) \quad \begin{aligned} f_\lambda &= f_r r_\lambda + f_t t_\lambda + f_p p_\lambda + f_q q_\lambda + f_u u_\lambda + f_x x_\lambda + f_y y_\lambda \\ &= f_r r_\lambda + f_t t_\lambda + [f]_x\, x_\lambda + [f]_y\, y_\lambda \,, \end{aligned}$$

while

$$(6.51)\quad f_\mu = f_r r_\mu + f_t t_\mu + f_p p_\mu + f_q q_\mu + f_u u_\mu + f_x x_\mu + f_y y_\mu$$
$$= f_r r_\mu + f_t t_\mu + [f]_x x_\mu + [f]_y y_\mu$$
$$+ f_p \psi_4 + f_q \psi_5 + f_u \psi_3 .$$

Thus by (6.45), (6.50) and (6.51),

$$(6.52)\quad \psi_{4,\lambda} - \varphi_{4,\mu} = r_\mu x_\lambda + f_\mu y_\lambda - r_\lambda x_\mu - f_\lambda y_\mu$$
$$= y_\lambda \left\{ f_p \psi_4 + f_q \psi_5 + f_u \psi_3 \right\}$$
$$+ \left(\frac{1+\delta}{2}\right) x_\lambda \psi_2 - \left(\frac{1+\delta}{2}\right) \rho\, y_\mu \varphi_2,$$

and

$$(6.53)\quad \psi_{5,\lambda} - \varphi_{5,\mu} = f_\mu x_\lambda + t_\mu y_\lambda - f_\lambda x_\mu - t_\lambda y_\mu$$
$$= x_\lambda \left\{ f_p \psi_4 + f_q \psi_5 + f_u \psi_3 \right\}$$
$$- \left(\frac{1+\delta}{2}\right) \sigma x_\lambda \psi_2 + \left(\frac{1+\delta}{2}\right) y_\mu \varphi_2 .$$

Taking into account the fact that System A is satisfied, we reduce (6.49), (6.52) and (6.53) to the system

$$(6.54)\quad \begin{aligned} \psi_{3,\lambda} &= \psi_4 x_\lambda + \psi_5 y_\lambda \\ \psi_{4,\lambda} &= y_\lambda \left\{ f_u \psi_3 + f_p \psi_4 + f_q \psi_5 \right\} \\ \psi_{5,\lambda} &= x_\lambda \left\{ f_u \psi_3 + f_p \psi_4 + f_q \psi_5 \right\} \end{aligned}$$

For fixed μ, (6.54) represents a system of linear, homogeneous, first order ordinary differential equations for the functions ψ_3, ψ_4 and ψ_5 of the variable λ. Moreover, by (6.43),

the homogeneous one point boundary conditions

$$\psi_3(0,\mu) = \psi_4(0,\mu) = \psi_5(0,\mu) = 0$$

must be satisfied. Hence, the unique solution for the system (6.54) is

$$\psi_3 = \psi_4 = \psi_5 = 0$$

wherever the solution of system A is defined.

Consider the linear algebraic system,

$$(6.55)\quad \begin{cases} \varphi_3 = u_\lambda - px_\lambda - qy_\lambda = 0 \\ \psi_3 = u_\mu - px_\mu - qy_\mu = 0\,. \end{cases}$$

The determinant of this system, by (6.20), does not vanish in a neighborhood of the origin, hence in this neighborhood there exists a unique solution for p and q. Since $p = u_x$ and $q = u_y$ satisfy (6.55) they are the solution of (6.55)

Similarly, from

$$(6.56)\quad \begin{cases} \varphi_4 = p_\lambda - r\,x_\lambda - fy_\lambda \\ \psi_4 = p_\mu - r\,x_\mu - fy_\mu\,, \end{cases}$$

we obtain $r = u_{xx}$ and $f = u_{xy}$,

while from

$$(6.57)\quad \begin{cases} \varphi_5 = q_\lambda - fx_\lambda - ty_\lambda \\ \psi_5 = q_\mu - fx_\mu - ty_\mu\,, \end{cases}$$

we obtain the additional information that $t = u_{yy}$. Consequently, any solution of System A under the given characteristic initial conditions satisfies the equation

$$u_{xy} = f(x,y; u; u_x, u_y; u_{xx}, u_{yy})$$

in a neighborhood of the point (0,0; 0; 0,0; 0,0) and the proof of Theorems 9 and 9a is now complete.

Let us designate the problem considered in Theorems 9 and 9a as Problem I. By virtue of the exposition of Chapter IV and this present chapter, we may associate to this problem a particular Problem II, of the type considered in Theorems 3 and 3a of Chapter II. As we have shown, any solution of I is a solution of II, and, conversely, any solution of II is a solution of I. Where for I, $F \in C'''$, Theorem 3 tells us that the solution of the related Problem II is unique. Hence, as is stated in Theorem 9, the solution for I is likewise unique. If, however, for Problem I, $F \in C''$ only, then Theorem 3a tells us merely that the related Problem II has at least one solution. Moreover, Example 1, Chapter II, tells us that this solution cannot be shown to be unique.

We must not conclude merely from the above that for $F \in C''$ the solution to Problem I cannot be shown to be unique. We can say, though, that any proof for uniqueness, if such can be made at all, will apparently have to be based upon arguments independent of those of this paper.

Chapter VII

The Mixed Boundary Value Problem for $u_{xy} = f(x,y;\ u;\ u_x,\ u_y)$.

In the terminology of J. HADAMARD [11], appendix II, p. 456, the mixed hyperbolic boundary value problem is one in which we prescribe the values of the integral surface along two lines issuing from a point, one of which is characteristic to the surface in question, while the other is nowhere characteristic.

J. HADAMARD, in the reference above, and E. PICARD [7], p.135, prove the existence of a unique solution to the linear equation

$$u_{xy} = a\,u_x + b\,u_y + c\,u\ , \tag{7.1}$$

a, b and c continuous functions of x and y alone, satisfying the initial conditions

$$u(x,0) = u(x,x) = 0. \tag{7.2}$$

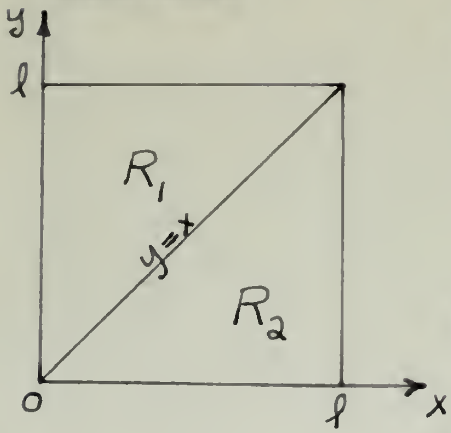
In Theorem 10, below, we extend their conclusions to the equation

$$u_{xy} = f(x,y;\ u;\ u_x,u_y) \tag{7.3}$$

maintaining initial conditions (7.2). The result is well known, but does not appear in the literature in the precise form stated. We require this precise statement because we wish to proceed from Theorem 10 by the methods of Chapters II and III in which we relax the Lipschitz condition on the function f to require merely

that f be partially Lipschitzian. Thus we obtain the improved statement of Theorem 10a.

Theorem 10



1) $f(x,y;\ u;\ p,q) \in C(B)$, $B: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \\ -a \leq u \leq a \\ -b \leq p \leq b \\ -b \leq q \leq b \end{cases}$

2) f is Lipschitzian on B (as defined in Theorem 1.)

3) $M\ell^2 \leq a$, $M\ell \leq b$, where $M = \max |f|$ on B

4) There exists one and only one function $u(x,y) \in C'(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \end{cases}$, such that for each $(x,y) \in R$, the point $(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)) \in B$, and

$$u_{xy}(x,y) = f(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)),$$
$$u(x,0) = u(x,x) = 0 \quad \text{for each } (x,y) \in R.$$

Proof

This proof is based upon PICARD's variation of the method of successive approximations, [1] p. 359 or [7] p. 117. Here the uniform convergence of the approximating functions to the solution is verified by means of a majorant series. The majorant series used is that obtained from the approximating functions converging uniformly to the solution for the particular linear equation

$$(7.4) \qquad W_{xy} = K\,(W + W_x + W_y)$$

with the same initial conditions. K is the Lipschitz constant for the function f of (7.3). PICARD applied this technique to the characteristic initial value problem, obtaining Theorem 1 of Chapter II. He thus obtained the theorem for the characteristic initial value problem for the non-linear equation (7.3) from the theorem for the characteristic initial value problem for the linear equation (7.1).

For the mixed boundary value problem under consideration, a curious situation arises. We do not obtain a majorant series from equation (7.4) under mixed initial conditions. However, we do find that PICARD's majorant series for the characteristic initial value problem serves as well for this problem. Thus Theorem 10 follows not from the theorem for the mixed boundary value problem for the linear equation (7.1) but from the theorem for the characteristic initial value problem for equation (7.1).

It is sufficient, as we shall demonstrate later, to show existence of a unique solution in region $R_2 : \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq x \end{cases}$. Assuming $(x,y) \in R_2$, we may express the problem as the integral equation

$$(7.5) \qquad u(x,y) = \int_y^x d\xi \int_0^y f(\xi, \eta ; u; u_x, u_y)\,d\eta .$$

By differentiation,

$$(7.6) \qquad u_x(x,y) = \int_0^y f(x, \eta ; u; u_x, u_y)\,d\eta ,$$

and

$$(7.7)\quad u_y(x,y) = \int_y^x f(\xi, y; u; u_x, u_y)\,d\xi - \int_0^y f(y, \eta; u; u_x, u_y)\,d\eta .$$

We form the successive approximations

$$(7.8)\quad \left\{ \begin{array}{l} u_1(x,y) = \int_y^x d\xi \int_0^y f(\xi, \eta; 0; 0,0)\,d\eta \\ u_2(x,y) = \int_y^x d\xi \int_0^y f(\xi, \eta; u_1; u_{1,x}, u_{1,y})\,d\eta \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ u_n(x,y) = \int_y^x d\xi \int_0^y f(\xi, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y})\,d\eta \\ \qquad \vdots \qquad\qquad\qquad \vdots \end{array} \right.$$

where, by differentiation,

$$(7.9)\quad u_{n,x}(x,y) = \int_0^y f(x, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y})\,d\eta , \qquad (n = 1,2,\cdots),$$

$$(7.10)\quad u_{n,y}(x,y) = \int_y^x f(\xi, y; u_{n-1}; u_{n-1,x}, u_{n-1,y})\,d\xi - \int_0^y f(y, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y})\,d\eta , \qquad (n = 1,2,\cdots,).$$

Since the point $(x,y; 0; 0,0) \in B$ for $(x,y) \in R_2$, by hypothesis 3),

$$\begin{aligned} |u_1(x,y)| &\leq M\, |x-y| \cdot |y| \leq M \ell^2 \leq a, \\ |u_{1,x}(x,y)| &\leq M\, |y| \leq M \ell \leq b, \\ |u_{1,y}(x,y)| &\leq M \left\{ |x-y| + |y| \right\} \\ &= M|x| \leq M \ell \leq b \end{aligned}$$

Thus, by induction, for all n and for any $(x,y) \in R_2$

$$(7.11)\quad \left\{ \begin{array}{l} |u_n(x,y)| \leq M\ell^2 \leq a, \\ |u_{n,x}(x,y)| \leq M\ell \leq b, \\ |u_{n,y}(x,y)| \leq M\ell \leq b. \end{array} \right.$$

Our purpose is to show that on R_2

(7.12) $\left\{u_n\right\} \xrightarrow{\text{unif}} u$, $\left\{u_{n,x}\right\} \xrightarrow{\text{unif}} u_x$ and $\left\{u_{n,y}\right\} \xrightarrow{\text{unif}} u_y$

such that the function u and its derivatives satisfy conclusion 4) for $(x,y) \in R_2$. To accomplish this we consider the successive approximations

(7.13)
$$\begin{aligned}
w_1(x,y) &= \int_0^x d\xi \int_0^y M \, d\eta \\
w_2(x,y) &= \int_0^x d\xi \int_0^y K(w_1 + w_{1,x} + w_{1,y}) d\eta \\
&\vdots \qquad\qquad \vdots \\
w_n(x,y) &= \int_0^x d\xi \int_0^y K(w_{n-1} + w_{n-1,x} + w_{n-1,y}) d\eta \\
&\vdots \qquad\qquad \vdots
\end{aligned}$$

where, by differentiation,

(7.14) $$w_{n,x}(x,y) = \int_0^y K\left[w_{n-1} + w_{n-1,x} + w_{n-1,y}\right](x,\eta)d\eta, \quad (n = 1,2,\cdots),$$

(7.15) $$w_{n,y}(x,y) = \int_0^x K\left[w_{n-1} + w_{n-1,x} + w_{n-1,y}\right](\xi,y)d\xi, \quad (n = 1,2,\cdots).$$

Here $M = \max |f|$ on R while K is the Lipschitz constant of hypothesis 2).

Now $w_1(x,y) = Mxy$, hence $w_1(x,y) = w_1(y,x)$. Moreover, $w_{1,x}(x,y) = My$, $w_{1,y}(x,y) = Mx$, hence $w_{1,x}(x,y) = w_{1,y}(y,x)$.

Let us make the inductive hypothesis that for some fixed positive integer n,

(7.16) $w_n(x,y) = w_n(y,x)$, $w_{n,x}(x,y) = w_{n,y}(y,x)$.

But this implies that

$$(7.17)\quad \left[w_n + w_{n,x} + w_{n,y}\right](x,y) = \left[w_n + w_{n,x} + w_{n,y}\right](y,x)$$

and thus, by (7.13),

$$w_{n+1}(x,y) = w_{n+1}(y,x).$$

Also, by (7.14) and (7.15), (7.17) implies that

$$\begin{aligned} w_{n+1,x}(x,y) &= \int_0^y K\left[w_n + w_{n,x} + w_{n,y}\right](x,\eta)\,d\eta \\ &= \int_0^y K\left[w_n + w_{n,x} + w_{n,y}\right](\xi,x)\,d\xi \\ &= w_{n+1,y}(y,x). \end{aligned}$$

Hence, by induction, (7.16) holds for $n = 1,2,\cdots$.

PICARD, in the references quoted above, shows that

$$(7.18)\quad \sum_{n=1}^{\infty} w_n = w, \quad \sum_{n=1}^{\infty} w_{n,x} = w_x, \quad \sum_{n=1}^{\infty} w_{n,y} = w_y,$$

each uniformly convergent on R, where the function w and its derivatives satisfy

$$(7.19)\begin{cases} w_{xy} = K(w + w_x + w_y), \\ w(x,0) = w(0,y) = 0. \end{cases}$$

We now show that these series are majorant to the series

$$(7.20)\quad \sum_{n=1}^{\infty} (u_n - u_{n-1}), \quad \sum_{n=1}^{\infty} (u_{n,x} - u_{n-1,x}), \quad \sum_{n=1}^{\infty} (u_{n,y} - u_{n-1,y}),$$

respectively, for each $(x,y) \in R_2$, (with $u_0 = 0$).

Now, for $(x,y) \in R_2$,

$$|u_1(x,y)| \le \int_y^x d\xi \int_0^y |f(\xi,\eta;\,0;\,0,0)|\,d\eta \le \int_0^x d\xi \int_0^y M\,d\eta = w_1(x,y)$$

$$|u_{1,x}(x,y)| \le \int_0^y |f(x,\eta;0;0,0)|\,d\eta \le \int_0^y M\,d\eta = w_{1,x}(x,y)$$

$$|u_{1,y}(x,y)| \leq \int_y^x |f(\xi, y; 0; 0,0)|\, d\xi + \int_0^y |f(y, \eta; 0; 0,0)|\, d\eta$$

$$\leq \int_y^x M d\xi + \int_0^y M d\eta$$

$$= \int_0^x M d\xi = w_{1,y}(x,y).$$

Also, abbreviating our notation somewhat,

$$|u_2 - u_1| \leq \int_y^x d\xi \int_0^y |f(\xi, \eta; u_1; u_{1,x}, u_{1,y}) - f(\xi, \eta; 0; 0,0)|\, d\eta$$

$$\leq \int_y^x d\xi \int_0^y K\left[|u_1| + |u_{1,x}| + |u_{1,y}|\right](\xi, \eta) d\eta$$

$$\leq \int_0^x d\xi \int_0^y K\left[w_1 + w_{1,x} + w_{1,y}\right](\xi, \eta) d\eta$$

$$= w_2,$$

$$|u_{2,x} - u_{1,x}| \leq \int_0^y K\left[w_1 + w_{1,x} + w_{1,y}\right](x, \eta) d\eta = w_{2,x}$$

$$|u_{2,y} - u_{1,y}| \leq \int_y^x K\left[w_1 + w_{1,x} + w_{1,y}\right](\xi, y) d\xi$$

$$+ \int_0^y K\left[w_1 + w_{1,x} + w_{1,y}\right](y, \eta) d\eta$$

$$= \int_y^x K\left[w_1 + w_{1,x} + w_{1,y}\right](\xi, y)\, d\xi$$

$$+ \int_0^y K\left[w_1 + w_{1,x} + w_{1,y}\right](\xi, y) d\xi$$

$$= \int_0^x K\left[w_1 + w_{1,x} + w_{1,y}\right](\xi, y) d\xi$$

$$= w_{2,y}.$$

Hence, by induction, we obtain for $n = 1, 2, \cdots$

$$|u_n - u_{n-1}| \leq w_n, \quad |u_{n,x} - u_{n-1,x}| \leq w_{n,x},$$

(7.21) $\quad |u_{n,y} - u_{n-1,y}| \leq w_{n,y}$ for each $(x,y) \in R_2$.

Thus the series of (7.18) are majorant to the corresponding series of (7.20). Moreover, the requirements for termwise differentiation of an infinite sum are satisfied since each of the series of (7.20) is now known to be uniformly convergent on R_2. Hence, for $(x,y) \in R_2$,

$$(7.22) \quad \begin{cases} \sum_{n=1}^{\infty} (u_n - u_{n-1}) = u \\ \sum_{n=1}^{\infty} (u_{n,x} - u_{n-1,x}) = u_x \\ \sum_{n=1}^{\infty} (u_{n,y} - u_{n-1,y}) = u_y \ ; \end{cases}$$

or, in other terms, since each of these series telescopes,

$$(7.22)' \quad \left\{u_n\right\} \xrightarrow{\text{unif}} u, \quad \left\{u_{n,x}\right\} \xrightarrow{\text{unif}} u_x, \quad \left\{u_{n,y}\right\} \xrightarrow{\text{unif}} u_y$$

on R_2.

We now verify that the function u and its derivatives u_x and u_y satisfy the integral equation statement of the problem (7.3):

$$\begin{aligned} &\left| u(x,y) - \int_y^x d\xi \int_0^y f(\xi, \eta ; u ; u_x, u_y) d\eta \right| \\ &\quad \leq \left| u(x,y) - u_n(x,y) \right| + \int_y^x d\xi \int_0^y \Big| f(\xi, \eta ; u ; u_x, u_y) \\ (7.23) \qquad &\qquad\qquad - f(\xi, \eta ; u_{n-1} ; u_{n-1,x}, u_{n-1,y}) \Big| d\eta \\ &\quad \leq \left| u(x,y) - u_n(x,y) \right| \\ &\qquad + \int_y^x d\xi \int_0^y K \Big[|u - u_{n-1}| + |u_x - u_{n-1,x}| + |u_y - \\ &\qquad\qquad u_{n-1,y}| \Big] (\xi, \eta) d\eta \end{aligned}$$

Thus, by (7.22)', given $\epsilon > 0$, there exists a positive integer N, depending on ϵ alone, such that $n > N \Rightarrow$

$$\left| u(x,y) - \int_y^x d\xi \int_0^y f(\xi, \eta; u; u_x, u_y) d\eta \right| < \epsilon (1 + 3K\ell^2),$$

for $(x,y) \in R_2$. But ϵ is arbitrary, hence the integral equation is satisfied.

By (7.11) and (7.22)' we see that for any $(x,y) \in R_2$, the point $(x, y; u(x,y); u_x(x,y), u_y(x,y)) \in B$. Thus existence of a solution on R_2 is now proved.

To prove uniqueness, let us suppose that u_1 and u_2 are two solutions on R_2, then

$$\begin{aligned} |u_1(x,y) - u_2(x,y)| &\leq \int_y^x d\xi \int_0^y | f(\xi, \eta; u_1; u_{1,x}, u_{1,y}) \\ &\qquad - f(\xi, \eta; u_2; u_{2,x}, u_{2,y}) | \, d\eta \\ &\leq \int_y^x d\xi \int_0^y K[\, |u_1 - u_2| + |u_{1,x} - u_{2,x}| + |u_{1,y} - u_{2,y}| \,] \\ &\qquad (\xi, \eta) d\eta, \end{aligned} \tag{7.24}$$

$$\begin{aligned} |u_{1,x}(x,y) - u_{2,x}(x,y)| &\leq \int_0^y | f(x, \eta; u_1; u_{1,x}, u_{1,y}) \\ &\qquad - f(x, \eta; u_2; u_{2,x}, u_{2,y}) | \, d\eta \\ &\leq \int_0^y K[\, |u_1 - u_2| + |u_{1,x} - u_{2,x}| + |u_{1,y} - u_{2,y}| \,](x, \eta) d\eta, \end{aligned} \tag{7.25}$$

$$\begin{aligned} |u_{1,y}(x,y) - u_{2,y}(x,y)| &\leq \int_y^x | f(\xi, y; u_1; u_{1,x}, u_{1,y}) \\ &\qquad - f(\xi, y; u_2; u_{2,x}, u_{2,y}) | \, d\xi \\ &\quad + \int_0^y | f(y, \eta; u_1; u_{1,x}, u_{1,y}) \\ &\qquad - f(y, \eta; u_2; u_{2,x}, u_{2,y}) | d\eta. \end{aligned} \tag{7.26}$$

Let $\psi(x,y) = [|u_1-u_2| + |u_{1,x}-u_{2,x}| + |u_{1,y}-u_{2,y}|](x,y)$.

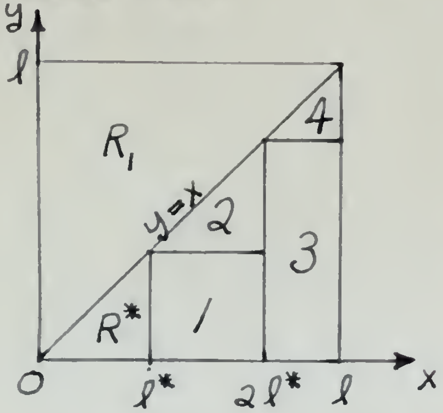
With $R^*: \begin{cases} 0 \leq x \leq \ell^* \\ 0 \leq y \leq x \end{cases}$, $\ell^* = \min(1, \ell, \frac{1}{6K})$, we have $\psi(x,y) \in C(R^*)$. Moreover, there exists a point $(x^*,y^*) \in R^*$ such that $\psi(x^*,y^*) = \mu$ where $\mu = \max \psi(x,y)$ on R^*. But, adding (7.24), (7.25) and (7.26) we obtain

$$\begin{aligned} \psi(x,y) &\leq K\,\mu\{(x-y)y + y + (x-y) + y\} \\ &\leq K\,\mu\,(xy + x + y) \\ &\leq K\,\mu \cdot \frac{3}{6K} = \frac{\mu}{2}, \end{aligned}$$

hence $\psi(x^*,y^*) = \mu \leq \frac{\mu}{2}$, which implies $\mu = 0$ and thus

$$u_1(x,y) = u_2(x,y) \tag{7.27}$$

for $(x,y) \in R^*$



To extend this uniqueness proof to the domain R_2, we subdivide R_2 as shown in the diagram. We know that the solution u is unique on R^* and hence determines $u(\ell^*,y)$ for $0 \leq y \leq \ell^*$.

But $u(x,0) = 0$ by hypothesis, consequently, by Theorem 1, Chapter II, we have a unique solution u_1 to the characteristic initial value problem on sub-region 1. Since $u_x(\ell^*,0) = u_{1,x}(\ell^*,0)$, we have from the differential equation that $u_x(\ell^*,y) = u_{1,x}(\ell^*,y)$ for $0 \leq y \leq \ell^*$, i.e. u and u_1 have a first order contact across the line $x = \ell^*$ and hence together represent a unique solution for the region $R^* + 1$. Analogously, by the preceding "in the

small" uniqueness proof for the mixed boundary value problem, the solution u_2 is unique in sub-region 2 and has a first order contact with u_1 across the line $y = \ell$. We continue obtaining unique solutions for characteristic initial value and mixed initial value problems, alternatively as indicated by the numerical sequence in the diagram. These solutions have first order contacts with each other across the characteristics forming the boundaries of the sub-regions, hence we have extended our uniqueness proof from the region R^* to the region R_2.

Having thus determined the existence of a unique solution satisfying conclusion 4) throughout R_2, we now consider the Cauchy problem for region R_1 with the same equation and hypotheses thereon and with the initial conditions

$$(7.28)\begin{cases} u^0(x,x) = 0,\ u^0_x(x,x) = u_{x+}(x,x), \text{ and} \\ u^0_y(x,x) = u_{y-}(x,x) \quad \text{for } x \in [0,\ell]. \end{cases}$$

In (7.28) u_{x+} and u_{y-} are the right-hand x and lower y derivatives, respectively, determined at each point of the line $y = x$ by the known solution u on R_2. By Theorem 4, Chapter III, there exists a unique solution u^0 to this Cauchy problem for each $(x,y) \in R_1$, hence

$$u_1(x,y) = \begin{cases} u_0(x,y) \text{ for } (x,y) \in R_1 \\ u(x,y) \text{ for } (x,y) \in R_2 \end{cases}$$

is the unique solution valid for each $(x,y) \in R = R_1 + R_2$, since u_0 and u have, by prescription, a first order contact across the line $y = x$. This completes the proof of Theorem 10.

Relaxing only hypothesis 2) of Theorem 10, we obtain the following improvement:

Theorem 10a

1)

2)' f is partially Lipschitzian on B (as defined in Theorem 1a.)

3)

⇒ 4)' There exists at least one function, etc. (as in Theorem 10.)

Outline of the proof:

As in the proof of Theorem 10, we may, without loss, prove existence on R_2 only. For, prescribing Cauchy conditions on $y = x$ as before, we may extend the solution from R_2 to R_1, by use of Theorem 4a, Chapter III.

In this proof we follow very closely the derivation of Theorem 1a, Chapter II; hence only the differences between the two proofs will be noted.

WEIERSTRASS' theorem tells us that there exists a sequence of polynomials, $\{g_\lambda\}$, converging uniformly to f on B. We extend the g_λ, $(\lambda = 1,2,\cdots)$, and f from B to

$$B': \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \gamma \\ -\infty < u < \infty \\ -\infty < p < \infty \\ -\infty < q < \infty \end{cases}$$

by definitions analogous to (2.1). There exists a constant $L > 0$ such that $|g_\lambda| \leq L$ in B' and for all λ. More-

over, the g_λ are "fully" Lipschitzian in B'. Hence by Theorem 10, (with $a \to \infty$, $b \to \infty$), for each g_λ there exists a unique function u_λ such that for $(x,y) \in R_2$

$$(7.29) \quad u_\lambda = \int_y^x d\xi \int_0^y g_\lambda(\xi, \eta ; u_\lambda ; u_{\lambda,x}, u_{\lambda,y}) d\eta ,$$

and thus

$$(7.30) \quad u_{\lambda,x} = \int_0^y g_\lambda(x, \eta ; u_\lambda ; u_{\lambda,x}, u_{\lambda,y}) d\eta ,$$

$$(7.31) \quad u_{\lambda,y} = \int_y^x g_\lambda(\xi, y ; u_\lambda ; u_{\lambda,x}, u_{\lambda,y}) d\xi - \int_0^y g_\lambda(y, \eta ; u_\lambda ; u_{\lambda,x}, u_{\lambda,y}) d\eta .$$

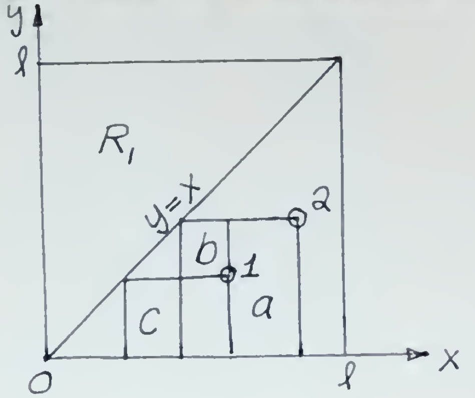
For $(x,y) \in R_2$, by (7.29), (7.30) and (7.31),

$$(7.32) \quad \left.\begin{aligned} |u_\lambda(x,y)| &\le L\ell^2 \\ |u_{\lambda,x}(x,y)| &\le L\ell \\ |u_{\lambda,y}(x,y)| &\le L\{(x-y) + y\} \\ &\le L\ell \end{aligned}\right\} (\lambda = 1, 2, \cdots)$$

i.e. the sequences $\{u_\lambda\}$, $\{u_{\lambda,x}\}$ and $\{u_{\lambda,y}\}$ are uniformly bounded on R_2.

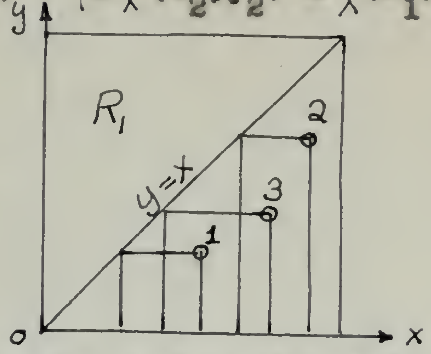
Given two points, $(x_1, y_1) \in R_2$, $(x_2, y_2) \in R_2$, we may assume, without loss, that $x_1 \le x_2$. Then, if $y_1 \le y_2$, let us assume that $y_2 < x_1$. Then by integrating over the regions a, b and c in diagram (A) we obtain

(A)



(7.33) $\quad |u_\lambda(x_2,y_2)-u_\lambda(x_1,y_1)| \le L\{\ell(x_2-x_1) + 2\ell(y_2-y_1)\}.$

(B)



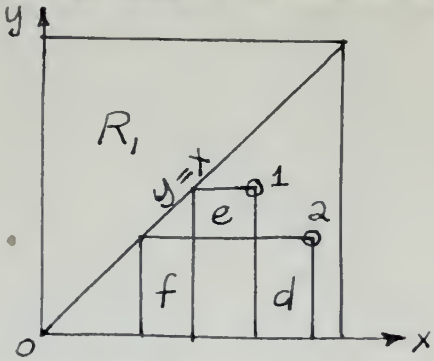
If $y_2 \geqslant x_1$ we may always choose a point (x_3,y_3) with $y_2 < x_3 < x_2$ and $y_1 < y_3 < x_1$ (as in diagram(B)). Then, as above,

$$|u_\lambda(x_2,y_2)-u_\lambda(x_3,y_3)| \le L\{\ell(x_2-x_3) + 2\ell(y_2-y_3)\}$$

$$|u_\lambda(x_3,y_3)-u_\lambda(x_1,y_1)| \le L\{\ell(x_3-x_1) + 2\ell(y_3-y_1)\}.$$

Adding, we obtain (7.33). Further if $y_1 \geqslant y_2$, we have the case shown in diagram (C). Here by integrating over the regions d, e and f we again obtain (7.33). Hence the sequence $\{u_\lambda\}$ is equicontinuous on R_2.

(C)



Now, for $(x,y_2) \in R_2$, $(x,y_1) \in R_2$, by (7.30)

(7.34) $\quad |u_{\lambda,x}(x,y_2)-u_{\lambda,x}(x,y_1)| \le L|y_2-y_1|.$

Likewise, for $(x_2,y) \in R_2$, $(x_1,y) \in R_2$, by (7.31)

(7.35) $\quad |u_{\lambda,y}(x_2,y) - u_{\lambda,y}(x_1,y)| \le L|x_2 - x_1|.$

Moreover, by precisely the same argument as that used to prove Lemma 2 of Chapter II, given $\mu > 0$, $\zeta > 0$, there exist $\delta > 0$, $N > 0$, depending only on μ and ζ, respectively, such that for $(x_2,y) \in R_2$, $(x_1,y) \in R_2$,

$$\lambda > N \text{ and } |x_2 - x_1| < \delta$$

$\Longrightarrow$

$$(7.36)\quad |u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y)| \leq K\int_0^y |u_{\lambda,x}(x_2,\eta) - u_{\lambda,x}(x_1,\eta)|\,d\eta + \mu + \zeta .$$

Thus by (.734), (7.36) and Lemma 1, Chapter II, the sequence $\{u_{\lambda,x}\}$ is equicontinuous on R_2.

We need the following refinement of the argument in order to show that the sequence $\{u_{\lambda,y}\}$ is equicontinuous on R_2:

Let us suppose $(x,y_2) \in R_2$, $(x,y_1) \in R_2$. Without loss, we may assume that $x \geqslant y_2 \geqslant y_1$. Then

$$u_{\lambda,y}(x,y_2) - u_{\lambda,y}(x,y_1)$$

$$= \int_{y_2}^{x} [g_\lambda(\xi,y_2;u_\lambda;u_{\lambda,x},u_{\lambda,y}) - g_\lambda(\xi,y_1;u_\lambda;u_{\lambda,x},u_{\lambda,y})]\,d\xi$$

$$(7.37)\qquad -\int_{y_1}^{y_2} g_\lambda(\xi,y_1;u_\lambda;u_{\lambda,x},u_{\lambda,y})\,d\xi$$

$$-\int_0^{y_1} [g_\lambda(y_2,\eta;u_\lambda;u_{\lambda,x},u_{\lambda,y}) - g_\lambda(y_1,\eta;u_\lambda;u_{\lambda,x},u_{\lambda,y})]\,d\eta$$

$$-\int_{y_1}^{y_2} g_\lambda(y_2,\eta;u_\lambda;u_{\lambda,x},u_{\lambda,y})\,d\eta$$

We have just proved that the sequences $\{u_\lambda\}$ and $\{u_{\lambda,x}\}$ are equicontinuous on R_2. The sequence $\{g_\lambda\}$ is certainly equicontinuous on E'. Hence, considering (7.35), given $\mu > 0$, there exists $\delta > 0$, depending upon μ alone, such that $|y_2 - y_1| < \delta$

$\Longrightarrow$

$$(7.38)\quad \left|\int_0^{y_1} [g_\lambda(y_2,\eta;u_\lambda;u_{\lambda,x},u_{\lambda,y}) - g_\lambda(y_1,\eta;u_\lambda;u_{\lambda,x},u_{\lambda,y})]\,d\eta\right| < \mu,$$

$$(7.39)\quad \Big|\int_{y_2}^{x} [g_\lambda(\xi,y_2;u_\lambda(\xi,y_2);u_{\lambda,x}(\xi,y_2),\ \underline{u_{\lambda,y}(\xi,y_2)})$$
$$-g_\lambda(\xi,y_1;u_\lambda(\xi,y_1);u_{\lambda,x}(\xi,y_1),\underline{u_{\lambda,y}(\xi,y_2)})]\,d\xi\Big| < \mu,$$

for $\lambda = 1,2,\cdots$.

Also, since $\{g_\lambda\} \xrightarrow{\text{unif}} f$ on E', given $\zeta > 0$, there exists $N > 0$, depending upon ζ alone, such that $\lambda > N$

$\Longrightarrow$

$$(7.40)\quad \left|\int_{y_2}^{x} [g_\lambda - f](\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda,x}(\xi, y_1), \underline{u_{\lambda,y}(\xi, y_2)})d\xi\right| < \zeta \quad ,$$

$$\left|\int_{y_2}^{x} [f-g_\lambda](\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda,x}(\xi, y_1), \underline{u_{\lambda,y}(\xi, y_1)})d\xi\right| < \zeta .$$

By hypothesis 2)',

$$(7.41)\quad \left|\int_{y_2}^{x} [f(\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda,x}(\xi, y_1), \underline{u_{\lambda,y}(\xi, y_2)}) - f(\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda,x}(\xi, y_1), \underline{u_{\lambda,y}(\xi, y_1)})]d\xi\right|$$

$$\leq \int_{y_2}^{x} K\,|u_{\lambda,y}(\xi, y_2) - u_{\lambda,y}(\xi, y_1)|\,d\xi .$$

Moreover, since $|g_\lambda| \leq L$, $(\lambda = 1,2,\cdots)$,

$$(7.42)\quad \left|\int_{y_1}^{y_2} g_\lambda(\xi, y_1; u_\lambda; u_{\lambda,x}, u_{\lambda,y})d\xi\right| \leq L\,|y_2 - y_1|$$

$$\left|\int_{y_1}^{y_2} g_\lambda(y_2, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y})d\eta\right| \leq L\,|y_2 - y_1| .$$

Thus by equations (7.37) through (7.41), given $\mu > 0$, $\zeta > 0$, there exists $\delta > 0$, $N > 0$, depending only upon μ and ζ, respectively, such that $|y_2 - y_1| < \delta$ and $\lambda > N$

$\Longrightarrow$

$$(7.43)\quad |u_{\lambda,y}(x,y_2) - u_{\lambda,y}(x,y_1)| \leq K\int_{y_2}^{x} |u_{\lambda,y}(\xi,y_2) - u_{\lambda,y}(\xi,y_1)|\, d\xi + 4\mu + 2\zeta .$$

By Lemma 1, Chapter II, inequalities (7.35) and (7.43) imply that the sequence $\{u_{\lambda,y}\}$ is equicontinuous on R_2.

From this point on the proof is practically identical with that for Theorem 1a. Since the sequences $\{u_\lambda\}$, $\{u_{\lambda,x}\}$ and $\{u_{\lambda,y}\}$ are uniformly bounded and equicontinuous on R_2, we may apply ARZELA's theorem to obtain a subsequence of each, uniformly convergent on R_2. Hence, as for Theorem 1a, by successive extractions of subsequences we obtain a subsequence $\{u^*_\lambda\}$ of $\{u_\lambda\}$ converging uniformly on R_2 to a solution u of the integral equation

$$u(x,y) = \int_y^x d\xi \int_0^y f(\xi,\eta;\, u;\, u_x,u_y)\, d\eta ,$$

and such that for $(x,y) \in R_2$ $(x,y;\, u(x,y);\, u_x(x,y),\, u_y(x,y)) \in B$. The proof for Theorem 10a is now complete.

Following E. PICARD [7] p. 135 and p. 139, we show that the general statement of the mixed boundary conditions, (i.e. where u is prescribed along two intersecting curves, one characteristic and the other nowhere characteristic), can be reduced to the statement found in Theorems 10 and 10a, (i.e. where $u(x,0) = u(x,x) = 0$ for $x \in [0,\ell]$).

First, let us suppose that we prescribe

$$(7.44) \qquad \begin{cases} u(x,0) = \varphi(x) \\ u(x,x) = \psi(x) \end{cases}$$

for $x \in [0,\ell]$, $\varphi(x)$ and $\psi(x) \in C'[0,\ell])$ and $\varphi(0) = \psi(0)$.

Consider

$$(7.45) \qquad w(x,y) = \varphi(x) + \psi(y) - \varphi(y).$$

We have $w_{xy} = 0$ on R while

$$(7.46) \qquad \begin{cases} w(x,0) = \varphi(x) \\ w(x,x) = \psi(x) \end{cases}$$

for $x \in [0,\ell]$. Hence, instead of the problem with non-homogeneous boundary conditions (7.44), by setting

$$(7.47) \qquad v = u - w$$

we may consider the problem

$$(7.48) \qquad \begin{cases} v_{xy} = f(x,y;\ v + w;\ v_x + w_x,\ v_y + w_y) \\ v\ (x,0) = 0 \\ v(x,x) = 0, \end{cases}$$

a problem of the type covered by Theorems 10 and 10a.

Second, suppose we prescribe u along the characteristic $y = 0$ and the nowhere characteristic curve $y = F(x)$, where $F(x) \in C'([0,\ell_1])$, $F'(x) \neq 0$ for $x \in [0,\ell_1]$ and $F(0) = 0$.

The coordinate transformation

$$(7.49) \qquad \begin{cases} \bar{x} = F(x) \\ \bar{y} = y \end{cases}$$

reduces the curve $y = F(x)$ to the diagonal $\bar{y} = \bar{x}$ since the inverse F^{-1} exists and is of class C' on $[0, F(\ell_1)]$. Moreover,

$$(7.50) \qquad u_{xy} = F'(x)\ u_{\overline{xy}}.$$

Since $F'(x) \neq 0$, the form of the differential equation remains unchanged and we reduce the problem to one with initial conditions in the form (7.44).

Thus the general statement of the mixed boundary value problem for

$$(7.3) \qquad u_{xy} = f(x,y;\ u;\ u_x,u_y)$$

can be reduced to the form treated in Theorems 10 and 10a. We note that whatever continuity and Lipschitz conditions are satisfied by (7.3) before transformation (7.49) and substitution (7.47) are satisfied as well after these operations are performed.

CHAPTER VIII

EXISTENCE THEOREMS BASED ON THE CONCEPT OF UPPER AND LOWER BOUNDING FUNCTIONS

For the ordinary differential equation $y' = f(x,y)$ with $y(x_0) = y_0$, O. PERRON [18], assuming f merely continuous, gives an existence proof that is entirely independent of the classical proofs and contains them as special cases. He bases his proof on the concept of under and over functions, defining $\varphi(x)$ to be an under function if $\varphi(x_0) = y_0$ and

$$(8.1) \qquad D_{\pm} \varphi(x) < f(x, \varphi(x))$$

and defining $\psi(x)$ to be an over function if $\psi(x_0) = y_0$ and

$$(8.2) \qquad D_{\pm} \psi(x) > f(x, \psi(x)).$$

The solutions are found to lie between the upper limit function g of the set of underfunctions and the lower limit function G of the set of overfunctions, g and G themselves being solutions.

M. MÜLLER [4] shows that PERRON's proof will not carry over directly to apply to a system.

$$(8.3) \qquad y_i = f_i(x,y_1,\cdots,y_n) \quad , \quad (i = 1,\cdots,n).$$

However, he is able to extend the classical theorem, obtaining a statement which is similar to that of PERRON and which reduces to the direct analogue of PERRON's theorem in the particular case where the functions f_i are monotonically increasing in the arguments $y_1,\cdots,y_n$.

In this chapter we return to the characteristic initial value problem for

$$u_{xy} = f(x,y;\ u;\ u_x,\ u_y). \tag{8.4}$$

We obtain results similar to those of MÜLLER above. In the following Theorems 11 and 11a we improve the statements of Theorems 1 and 1a, Chapter II, by the introduction of upper and lower bounding functions Ω and ω.

Theorem 11 (11a)

1) $f(x,y;\ u;\ p,q) \in C(T)$, $T: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \\ \omega(x,y) \leq u \leq \Omega(x,y) \\ \omega_x(x,y) \leq p \leq \Omega_x(x,y) \\ \omega_y(x,y) \leq q \leq \Omega_y(x,y) \end{cases}$

2) (2)') f is *Lipschitzian* (*partially Lipschitzian*) on T (as defined in Theorems 1 and 1a).

3) The functions $\omega(x,y)$ and $\Omega(x,y) \in C'(R)$, $R: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \end{cases}$ with $\omega_{xy}(x,y)$ and $\Omega_{xy}(x,y) \in C(R)$. Moreover,

$$\omega(x,0) = \Omega(x,0) = 0 \quad \text{for } x \in [0,\ell],$$

$$\omega(0,y) = \Omega(0,y) = 0 \quad \text{for } y \in [0,\ell],$$

and, for each $(x,y) \in R$,

$$\omega_{xy}(x,y) \leq \min_{S(x,y)} \left[f(x,y;\ u;\ p,q) \right], \tag{8.5}$$

$$\Omega_{xy}(x,y) \geq \max_{S(x,y)} \left[f(x,y;\ u;\ p,q) \right] \tag{8.6}$$

where

$$(8.7)\qquad t(x,y):\begin{cases} x = x \\ y = y \\ \omega(x,y) \leq u \leq \Omega(x,y) \\ \omega_x(x,y) \leq p \leq \Omega_x(x,y) \\ \omega_y(x,y) \leq q \leq \Omega_y(x,y) \end{cases}$$

$\Longrightarrow$ 4) (4)') There exists one and only one (at least one) function $u(x,y) \in C'(R)$, $u_{xy} \in C(R)$ such that for each $(x,y) \in R$ the point $(x,y;\ u(x,y);\ u_x(x,y)\ u_y(x,y)) \in T$, and

$$u_{xy}(x,y) = f(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)),$$
$$u(x,0) = u(0,y) = 0 \quad \text{for each } (x,y) \in R.$$

Proof

We extend the domain of definition of the function f over T to B': $\begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \\ -\infty < u < \infty \\ -\infty < p < \infty \\ -\infty < q < \infty \end{cases}$ by defining $f(x,y;\ u;\ p,q)$ $= f(x,y;\ \overline{u};\ \overline{p},\overline{q})$, where

$$\begin{aligned} &\overline{u} = u \text{ if } \omega(x,y) \leq u \leq \Omega(x,y), && \overline{p} = p \text{ if } \omega_x(x,y) \leq p \leq \Omega_x(x,y), \\ (8.8)\quad &\overline{u} = \omega(x,y) \text{ if } u < \omega(x,y) && \overline{p} = \omega_x(x,y) \text{ if } p < \omega_x(x,y) \\ &\overline{u} = \Omega(x,y) \text{ if } \Omega(x,y) < u && \overline{p} = \Omega_x(x,y) \text{ if } \Omega_x(x,y) < p \end{aligned}$$

and
$$\begin{aligned} \overline{q} &= q \text{ if } \omega_y(x,y) \leq q \leq \Omega_y(x,y) \\ \overline{q} &= \omega_y(x,y) \text{ if } q < \omega_y(x,y) \\ \overline{q} &= \Omega_y(x,y) \text{ if } \Omega_y(x,y) < q. \end{aligned}$$

By definition (8.8), f is uniformly continuous and uniformly bounded in B'. Moreover, by hypothesis 2)(2)') and (8.8) f satisfies a Lipschitz (partial Lipschitz) condition in B'.

Hence, by Theorem 1 (1a) Chapter II, there exists one and only one (at least one) function satisfying conclusion 4)(4)') except that for $(x,y) \in R$ we are assured only that the point $(x,y;u(x,y);u_x(x,y),u_y(x,y)) \in S'$. To complete the proof we must show that this point actually lies in T; i.e. we must show that for each $(x,y) \in R$,

$$(8.9)\quad \begin{cases} \omega(x,y) \leq u(x,y) \leq \Omega(x,y) \\ \omega_x(x,y) \leq u_x(x,y) \leq \Omega_x(x,y) \\ \omega_y(x,y) \leq u_y(x,y) \leq \Omega_y(x,y) \,. \end{cases}$$

To accomplish this, we first prove the following lemma:

Lemma 3 i) $\omega_{xy}(x,y) \leq u_{xy}(x,y)$ for all $(x,y) \in R$

$$\Longrightarrow \quad \omega(x,y) \leq u(x,y) \quad "$$

$$\omega_x(x,y) \leq u_x(x,y) \quad "$$

$$\omega_y(x,y) \leq u_y(x,y) \quad " \quad ,$$

ii) $\Omega_{xy}(x,y) \geq u_{xy}(x,y)$ for all $(x,y) \in R$

$$\Longrightarrow \quad \Omega(x,y) \geq u(x,y) \quad "$$

$$\Omega_x(x,y) \geq u_x(x,y) \quad "$$

$$\Omega_y(x,y) \geq u_y(x,y) \quad " \quad .$$

Proof: For i),

$$\omega(x,y) = \int_0^x dx \int_0^y \omega_{xy} dy \leq \int_0^x dx \int_0^y u_{xy} dy = u(x,y)$$

$$\omega_x(x,y) = \int_0^y \omega_{xy} dy \leq \int_0^y u_{xy} dy = u_x(x,y)$$

$$\omega_y(x,y) = \int_0^x \omega_{xy} dx \leq \int_0^x u_{xy} dx = u_y(x,y).$$

The proof for ii) is analogous.

To prove (2.9) it only remains to verify that hypothesis i) and ii) of Lemma 3 are satisfied by u. By hypothesis 3) and definition (2.8), for each $(x,y) \in R$,

$$\begin{aligned}\omega_{xy}(x,y) &\leq \min_{S(x,y)} \left[f(x,y;\, u;\, p,q) \right] \\ &\leq f(x,y;\, u(x,y);\, u_x(x,y),\, u_y(x,y)) \\ &= u_{xy}(x,y)\end{aligned}$$

and

$$\begin{aligned}\Omega_{xy}(x,y) &\geq \max_{S(x,y)} \left[f(x,y;\, u;\, p,q) \right] \\ &\geq f(x,y;\, u(x,y);\, u_x(x,y),\, u_y(x,y)) \\ &= u_{xy}(x,y).\end{aligned}$$

Thus, by Lemma 3, requirement (2.9) is satisfied for each $(x,y) \in R$ and the proof of Theorems 11 and 11a is complete.

It is evident upon inspection of Theorems 11 and 11a that if, instead of homogeneous initial conditions, we prescribe

$$u(x,0) = U(x) \quad \text{with } U(x) \in C'([0,\ell]),$$
$$u(0,y) = V(y) \quad \text{with } V(y) \in C'([0,\ell]),$$

where $U(0) = V(0)$, then we must require

$$\omega(x,0) = \Omega(x,0) = U(x),$$
$$\omega(0,y) = \Omega(0,y) = V(y).$$

The proof then goes through as before.

The following example is an illustration of Theorem 11:

Example 4

For the problem

(8.10) $u_{xy} = (2^{1/m} - u_x)^{1/m+1}$, $u(x,0) = u(0,y) = 0$,

we may readily verify that

(8.11) $\omega(x,y) = \left(\frac{1}{m+1}\right)^{1/m+1} \cdot 2^{1/m(m+1)}\, xy$

and

(8.12) $\Omega(x,y) = 2^{1/m(m+1)}\, xy$

satisfy the hypotheses of Theorem 11 for all $x \geqslant 0$ and

$$0 \leq y \leq C^*_m = \frac{m}{m+1} \quad 2^{1/m+1}$$

In Chapter II we obtained the exact solution

(2.42) $u(x,y) = x\left\{2^{1/m} - \left[\frac{m}{m+1}(C_m - y)\right]^{m+1/m}\right\}$

where

(2.43) $C_m = \frac{m+1}{m} \quad 2^{1/m+1}$

is a branch point of the solution. We observe that as m increases indefinitely ω and Ω approach u from below and above, respectively, while C^*_m approaches C_m from below.

We see from this example that it is possible to obtain approximate solutions, with known limits of error, and to locate singularities in the actual solution by use of Theorem 11, provided that suitable functions ω and Ω can be obtained. For problems where explicit solutions cannot be obtained in "closed form", the procedure is to alter the right-hand side of the equation

$$u_{xy} = f(x,y;\, u;\, u_x,\, u_y)$$

so that an explicit solution of the altered equation can be ob-

tained satisfying the boundary conditions. This may lead to functions ω and Ω satisfying the hypotheses of Theorem 11. (See W. M. WHYBURN [19] and [20].) The motivation for equations (3.11) and (3.12) of Example 4 is now evident.

When we consider the possibility of applying, as explained below, the PERRON method using under and over functions to the characteristic initial value problem under consideration, we find the situation much the same as that in the case of a system of first order ordinary differential equations. We arrive at the unsatisfactory state of affairs wherein there is no assurance that the under functions remain below the over functions throughout the entire region on which a solution is known to exist. In fact, we shall presently give an example where an under function exceeds an over function within the domain of existence of a solution.

Recalling inequalities (3.1) and (3.2), we may express the application of the PERRON method as follows: We require both the under and over functions to satisfy the given characteristic initial conditions and to be continuously differentiable and to possess a mixed second derivative at each point of the domain $R: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \end{cases}$. We further stipulate that each under function, , shall satisfy

$$(3.13) \quad \varphi_{xy}(x,y) < f(x,y;\ \varphi(x,y);\ \varphi_x(x,y),\ \varphi_y(x,y)),$$

and that each over function, ψ, shall satisfy

$$(3.14) \quad \psi_{xy}(x,y) > f(x,y;\ \psi(x,y);\ \psi_x(x,y),\ \psi_y(x,y))$$

for each $(x,y) \in R$.

Analogous arguments to those used by PERRON for the ordinary differential equation $y' = f(x,y)$ lead to the inequalities

$$\varphi_x(0,y) < \psi_x(0,y) \qquad \text{for} \qquad 0 < y \leq \ell ,$$
$$\varphi_y(x,0) < \psi_y(x,0) \qquad \text{for} \qquad 0 < x \leq \ell ,$$

for any under function φ and any over function ψ. These inequalities, together with the requirement that φ and ψ satisfy the characteristic initial data on the positive x and y axes, insure that $\psi > \varphi$ in a sufficiently small "L" shaped strip in the first quadrant adjacent to the initial characteristics. Unfortunately, this is inadequate as the following example demonstrates.

Example 5

Consider the problem

$$(8.15) \qquad u_{xy} = 0, \quad u(x,0) = u(0,y) = 0.$$

This problem has the unique solution $u \equiv 0$ throughout the finite plane. Let

$$(8.16) \begin{cases} \psi_{xy} = Ax - By^2 + C \\ \varphi_{xy} = -D , \end{cases}$$

where A, B, C and D are positive constants. By integration in (8.16) we may obtain functions ψ and φ satisfying the initial conditions of (8.15). Obviously, φ is an under function for all (x,y). Moreover, $\psi_{xy} > 0$ for all (x,y) lying in the portion of the first quadrant below the parabolic arc

$$y = +\sqrt{\frac{A}{B}x + \frac{C}{B}} \,;$$

and hence ψ meets the requirements for an over function on a domain R_ℓ: $\begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \sqrt{\frac{C}{B}} \end{cases}$ where ℓ is arbitrarily large but finite.

Defining $h = \psi - \varphi$ we have

$$h_{xy}(x,y) = Ax - By^2 + C + D.$$

Since $h(x,0) = h(0,y) = 0$, we obtain by integration

$$h(x,y) = \frac{A}{2} x^2y - \frac{B}{3} x^2y^2 + (C+D)\, xy\,.$$

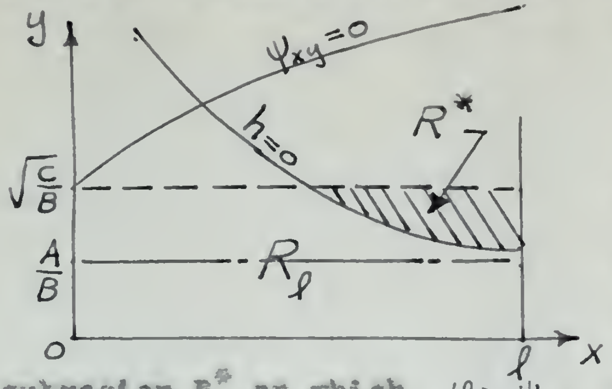
We note that $h > 0$ in that portion of the first quadrant below the hyperbola branch

$$y = \frac{A}{B} + \frac{2(C+D)}{Bx}$$

while $h < 0$ above this branch. From the diagram it is evident that if we require

$$\frac{A}{B} < \sqrt{\frac{C}{B}}$$

then there exists a positive constant ℓ such that within the corresponding domain R_ℓ we have a subregion R^* on which $\varphi > \psi$. Hence the PERRON method is not directly applicable to this class of problems.



Returning to Theorems 11 and 11a, we observe that if, for fixed (x,y), f is a monotonically increasing function for the arguments u, p and q, then

$$f(x,y;\ \omega(x,y);\ \omega_x(x,y),\ \omega_y(x,y)) = \min_{S(x,y)} \left[f(x,y;\ u;\ p,q) \right],$$

and

$$f(x,y;\ \Omega(x,y);\ \Omega_x(x,y),\ \Omega_y(x,y)) = \max_{S(x,y)} \left[f(x,y;u;p,q) \right].$$

In this case we may alter hypothesis 3) to require merely that

$$\omega_{xy}(x,y) \leqslant f(x,y;\ \omega(x,y);\ \omega_x(x,y),\ \omega_y(x,y))$$
$$\Omega_{xy}(x,y) \geqslant f(x,y;\ \Omega(x,y);\ \Omega_x(x,y),\ \Omega_y(x,y))$$

for each $(x,y) \in H$. This is the direct analogue to PERRON's theorem (see [15]) and corresponds to the previously mentioned result of MÜLLER for a system (6.3).

We close this chapter with the remark that Theorems 11 and 11a can be extended immediately in two ways. First, the method is directly applicable to the Cauchy problem. We require the functions ω and Ω to satisfy the Cauchy initial data and observe that the proof of Lemma 3 is essentially unchanged. Second, the method extends to apply to a system

$$u_{i,xy} = f_i(x,y;\ u_j;\ u_{j,x},\ u_{j,y}), \qquad (i = 1,\cdots,n)$$

for both characteristic and Cauchy initial value prescriptions. The modifications in the hypotheses and proof for Theorems 11 and 11a are obvious.

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ON THE EXISTENCE OF NOT NECESSARILY UNIQUE SOLUTIONS OF THE CLASSICAL HYPERBOLIC BOUNDARY VALUE PROBLEMS FOR NONLINEAR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS IN TWO INDEPENDENT VARIABLES.

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NOTATIONS

The following special notations will be used throughout this paper with the meanings as defined below. Other special notations used will be defined at the place where they are introduced.

$\in$	is a member of; i.e. belongs to.
$R: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \end{cases}$	R is the set of all ordered pairs (x,y), (points) for which $0 \leq x \leq \ell$ and $0 \leq y \leq \ell$.
$f \in C(B)$	f is a member of the class of functions continuous on the set B.
$g \in C'(H)$	g is a member of the class of functions continuously differentiable on the set H, (and similarly for higher degrees of differentiability.)
u_x	$\frac{\partial u}{\partial x}$.
$u_{\lambda,x}$	$\frac{\partial u_\lambda}{\partial x}$.
$\dot{x}$	$\frac{dx}{d\tau}$ where τ is a parameter along a path.
$x \in [0,\ell]$	x belongs to the closed interval, $0 \leq x \leq \ell$.
$\Longrightarrow$	implies.
$\Longleftrightarrow$	implies and is implied by; i.e. if and only if.
$\{g_\lambda\}(x,y;\ u;\ p,q)$	a sequence of functions g_λ, ($\lambda = 1,2,\cdots$), of arguments (x,y; u; p,q).
$\{g_\lambda\} \longrightarrow f$ on B	the sequence $\{g_\lambda\}$ converges pointwise on the set B to the function f.

$\{g_\lambda\} \xrightarrow{\text{unif}} f$ on B	the sequence $\{g_\lambda\}$ converges uniformly on the set B to the function f.
$D_\pm y$	the right(+) and left (-) hand derivatives of the function y at the point in question.

CHAPTER I

INTRODUCTION

The purpose of this paper is to present a number of existence theorems pertaining to a class of non-linear second order partial differential equations in two independent variables of the general form

$$(1.1)\qquad F(x,y;\ u;\ p,q;\ r,s,t) = 0,$$

where

$$(1.2)\qquad p = u_x,\quad q = u_y,\quad r = u_{xx},\quad s = u_{xy} \text{ and } t = u_{yy}\ ,$$

in the usual notation. We restrict our attention to those prescriptions of initial conditions for which integral surfaces exist such that the equation is of hyperbolic type thereon, i.e. the inequality

$$(1.3)\qquad F_s^2 - 4\,F_r\,F_t > 0$$

must be satisfied on the integral surface in a neighborhood of the initial data.

E. PICARD [1],[7][1], E. GOURSAT [8], E.E.Levi[9], H.LEWY[10], J. HADAMARD[11], M. CINQUINI-CIBRARIO[12],[13], and others have

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[1] The number in the bracket [ ] refers to the reference in the bibliography.

developed existence theorems based on the method of successive approximations. Their concern has been to establish sufficient conditions for the existence of a unique solution. Retaining their restrictions on the initial data, we shall obtain sufficient conditionsfor the existence of at least one solution. The integrals of the equations we consider will not, in general, be unique.

The concept of characteristic curves in an integral surface plays an important role in all work in this field. We give two definitions of a characteristic curve, the first applicable when the curve is expressed in non-parametric form, the second when expressed in parametric form:

Definition 1

$$\Upsilon : \begin{cases} a \leq x \leq b \\ y = g(x) \end{cases} \quad \text{where } g \in C'([a,b]), \text{ or } \quad \Upsilon : \begin{cases} x = h(y) \\ c \leq y \leq d \end{cases}$$

where $h \in C'([c,d])$, is a characteristic base curve (characteristic projection or, by usage, characteristic) for a particular integral surface J: $u=u(x,y)$ of $F(x,y;\ u;\ p,q;\ r,s,t) = 0 \Longleftrightarrow$ for each (x,y)

$$F_r dy^2 - F_s dydx + F_t dx^2 = 0 \tag{1.4}$$

Definition 1a

$\Upsilon : \begin{cases} x=x(\tau) \\ y=y(\tau) \end{cases}$ for $\tau \in [0,1]$ and where $x,y \in C'([0,1])$, is a characteristic base curve for a particular integral surface J: $u = u(x,y)$ of $F(x,y;\ u;\ p,q;\ r,s,t) = 0 \Longleftrightarrow$ for each $\tau \in [0,1]$

$$(1.5) \quad \begin{cases} 1) \quad F_r \dot{y}^2 - F_s \dot{y}\dot{x} + F_t \dot{x}^2 = 0 \\ 2) \quad \dot{x}^2 + \dot{y}^2 \neq 0. \end{cases}$$

Under either definition Υ is rectifiable and possesses a continuously turning tangent (see C. JORDAN[6], p. 100). The two definitions are equivalent in the following sense: We may convert Υ expressed in non-parametric form into its parametric expression by setting $x = \tau$, $y = g(\tau)$, or $x = h(\tau)$, $y = \tau$ as the case may be. That the converse is possible follows directly from condition 2) of Definition 1a and the Implicit Function Theorem. For, suppose at a point $(x(\tau_0), y(\tau_0))$ of Υ that $\dot{x} \neq 0$. Then in a vicinity of $x_0 = x(\tau_0)$ the inverse relation $\tau = \tau(x)$ exists and we may write

$$(1.6) \qquad \Upsilon \; : \; y = y(\tau(x)) = g(x).$$

Similarly, where $\dot{y} \neq 0$, we may write

$$(1.7) \qquad \Upsilon \; : \; x = x(\tau(y)) = h(y).$$

By condition 2), one of the two representations (1.6) or (1.7) is always possible in the vicinity of each point of Υ.

Definition 2

$$\Gamma \; : \; \begin{array}{l} x = x(\tau) \\ y = y(\tau) \\ u = u(\tau) \end{array} \text{ for } \tau \in [0,1] \text{ and where } x,y,u \in C'([0,1]),$$

a space curve lying in a particular integral surface J: $u=u(x,y)$ of $F(x,y;\, u;\, p,q;\, r,s,t) = 0$, is called a characteristic curve in the integral surface $J \Longleftrightarrow$ the projection of Γ onto the xy plane is a characteristic projection for the integral surface J.

Under suitable hypotheses, by virtue of the hyperbolic condition (1.3), for any integral surface J: $u=u(x,y)$ of $F(x,y;u;p,q;,r,s,t)=0$, equations (1.4) or (1.5) determine two one parameter families of characteristic curves lying in the integral surface J. Exactly one characteristic curve from each family passes through any given point $(x_0,y_0,u(x_0,y_0))$ of the integral surface J; and, moreover, the corresponding two characteristic base curves do not have a common tangent at (x_0,y_0).

Along any curve, characteristic or otherwise, lying in the integral surface J, the following strip, or band, conditions

$$\dot{u} = p\dot{x} + q\dot{y} \tag{1.8}$$

$$\begin{cases} \dot{p} = r\dot{x} + s\dot{y} \\ \dot{q} = s\dot{x} + t\dot{y} \end{cases} \tag{1.9}$$

must be satisfied.

The modification of Definition 2 and conditions (1.8),(1.9) when the curve Γ is expressed in non-parametric form is obvious.

Definition 3

$$S^1: \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \\ p=p(\tau) \\ q=q(\tau) \end{cases} \quad \text{for } \tau \in [0,1] \text{ and where } x,y,u,p,q \in C'([0,1]).$$

is called a first order strip $\Longleftrightarrow$ for each $\tau \in [0,1]$

$$\dot{u} = p\dot{x} + q\dot{y} \tag{1.8}$$

Suppose a particular integral surface J: $u=u(x,y)$ of

$F(x,y;\ u;\ p,q;\ r,s,t) = 0$ has a contact of first order with the strip S^1. Then if $\Gamma : \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \end{cases}$ for $\tau \in [0,1]$ is a characteristic curve in the integral surface J, the strip S^1 is called a characteristic first order strip for the integral surface J.

Definition 4

$$S^2: \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \\ p=p(\tau) \\ q=q(\tau) \\ r=r(\tau) \\ s=s(\tau) \\ t=t(\tau) \end{cases} \quad \text{for } \tau \in [0,1] \text{ and where } x,y,u,p,q,r,s,t \in C'([0,1])$$

is called a second order strip $\Longleftrightarrow$ for each $\tau \in [0,1]$

$$\dot{u} = p\dot{x} + q\dot{y} \tag{1.8}$$

$$\begin{cases} \dot{p} = r\dot{x} + s\dot{y} \\ \dot{q} = s\dot{x} + t\dot{y} \end{cases} \tag{1.9}$$

If, moreover, equation (1.1) and conditions (1.3) and (1.5) are satisfied for each $\tau \in [0,1]$, then S^1 is called a characteristic second order strip.

Note in Definition 4 that since all the arguments of the functions involved in conditions (1.5) are known upon prescription of the strip S^2, we may determine whether or not the projection of corresponding space curve $\Gamma : \begin{cases} x=x(\tau) \\ y=y(\tau) \\ u=u(\tau) \end{cases}$ for $\tau \in [0,1]$ is a characteristic projection without reference to any particular integral surface.

Definitions 3 and 4 can be readily modified to deal with the non-parametric case. See, for example, M. CINQUINI-CIBRARIO[13].

In Chapter 2 we consider the characteristic initial value problem for the equation

$$s = f(x,y;\ u;\ p,q) \tag{1.10}$$

and its extension to the system of equations

$$s_i = f_i(x,y;\ u_1,\cdots,u_n;\ p_1,\cdots,p_n,\ q_1,\cdots,q_n) \quad (i=1,2,\cdots,n). \tag{1.11}$$

We modify the customary hypothesis that f be Lipschitzian, i.e. with respect to variables u, p and q, to require that f be partially Lipschitzian, i.e. with respect to variables p and q only. We obtain existence of an integral u over the same closed domain as that obtained in the classical theory. Our integral, however, cannot be shown to be unique. This fact is demonstrated by an example. By further example, we show that the bounds obtained on the domain of existence are maximal bounds.

In Chapter 3 we apply the methods of Chapter 2 to the Cauchy problem for equation (1.10) and the extension to the system (1.11). The conclusions are similar to those obtained in Chapter 2.

The arguments in Chapter 4 serve to establish the equivalence (as defined therein) between the characteristic initial value and the Cauchy problems for the system (1.11) and the corresponding problems for a particular system of first order partial differential equations of the form

$$(1.12)\qquad \begin{cases} \sum_{k=1}^{n} A_{ik} U_{k,x} = C_i & (i = 1,2,\cdots,m < n) \\ \sum_{k=1}^{n} A_{ik} U_{k,y} = C_i & (i = m+1,m+2,\cdots,n) \end{cases}$$

where the A_{ik}, C_i are functions of $x, y, u_1, u_2, \cdots, u_n$. The system (1.12) is termed a canonical hyperbolic system.

This equivalence has already been established by M. CINQUINI-CIBRARIO[12]. Under the restriction that the first partial derivatives of the functions A_{ik}, C_i be Lipschitzian with respect to all their arguments, she obtains her theorems on the existence and uniqueness of the system of functions U_i as the solution for the canonical hyperbolic system (1.12). We demonstrate that her reasoning establishing the equivalence does not depend upon the uniqueness of the solutions for either system (1.11) or system (1.12). Consequently, from our results in Chapters 2 and 3, we are able to remove the above Lipschitz condition entirely and obtain existence, but not uniqueness, for the solutions of the canonical hyperbolic system for both characteristic and Cauchy initial value prescriptions.

Following the attack of H. LEWY[10], in Chapter 5 we reduce the equation

$$(1.1)\qquad F(x,y;\ u;\ p,q;\ r,s,t) = 0$$

to a system of so-called characteristic equations by means of a transformation to the characteristic base curves as coordinates. This system is shown to contain a canonical hyperbolic system.

We treat the Cauchy problem, i.e. to find an integral surface which has a second order contact with a prescribed second order strip. By virtue of a theorem by M. CINQUINI-CIBRARIO, stated in Chapter 4, LEWY'S work yields immediately the result that for $F \in C'''$ in a suitable region, there exists a unique solution $u \in C'''$ in a sufficiently small neighborhood of the initial curve. We again demonstrate that the equivalence of the problems is not dependent upon uniqueness of their respective solutions; and, hence, by requiring simply that $F \in C''$ we obtain existence but not uniqueness.

In Chapter 6 we treat the characteristic initial value problem for equation (1.1). We follow a modification of H. LEWY'S method introduced by M. CINQUINI-CIBRARIO[13]. Here equation (1.1) is first transformed into the form

$$s = f(x,y;\ u;\ p,q;\ r,t). \tag{1.13}$$

A modified system of characteristic equations is obtained. This system also contains a canonical hyperbolic system. The theorems of Chapter 2 apply and we obtain results similar to those obtained in Chapter 5 for the Cauchy problem.

In Chapter 7 we treat the mixed boundary value problem for the equation

$$s = f(x,y;\ u;\ p,q), \tag{1.10}$$

i.e. the problem where any integral surface of (1.10) is required to pass through two space curves issuing from a point, with one of the curves being a characteristic on this surface and the other

curve having curve having nowhere a characteristic projection. We show that for equation (1.10) there is no loss in generality if we assume the initial data to be

(1.14) $$u(x,0) = u(x,x) = 0.$$

For f continuous, bounded and Lipschitzian, we prove that there exists one and only one integral surface of (1.10) satisfying (1.14) on a domain for which we prescribe explicit bounds. For f continuous, bounded and partially Lipschitzian, we find, by arguments analogous to those used in Chapters 2 and 3, that there exists at least one integral surface of (1.10) satisfying (1.14) on a domain for which we again prescribe the same type of explicit bounds.

In Chapter 8 we consider the characteristic initial value problem for equation (1.10) from a new point of view. Here, in order to extend the theorems of Chapter 2, we introduce the concept of upper and lower bounding functions for the solution (or solutions) of the problem. This idea was first used by O. PERRON [18] to obtain an existence proof for the problem

(1.15) $$y' = f(x,y) \quad , \quad y(x_0) = y_0.$$

His proof is quite independent of the classical proofs.

M. MÜLLER [4] shows that PERRON's method has no direct analogue for a system

(1.16) $$y_i' = f_i(x,y_1,\cdots y_n) \quad , \quad (i = 1,\cdots,n).$$

He is able, however, to extend the classical theorem for a system (1.16) to obtain a theorem which reduces to the direct analogue to the PERRON theorem in the case where the f_i are monotonically increasing functions of the arguments $y_1,\cdots,y_n$.

The extensions to the theorems of Chapter 2 which we obtain are similar to MÜLLER's conclusions for the system (1.16). Moreover, we demonstrate by example that the PERRON method has no direct analogue for the characteristic initial value problem for equation (1.10). We also give an example illustrating the theorems obtained in this chapter. Finally, we note that the Cauchy problem for equation (1.10) and the Cauchy and characteristic initial value problems for the system

$$(1.11) \qquad s_i = f_i(x,y;\, u_1,\cdots,u_n;\, p_1,\cdots,p_n,\, q_1,\cdots q_n) \qquad (i = 1,\cdots,n),$$

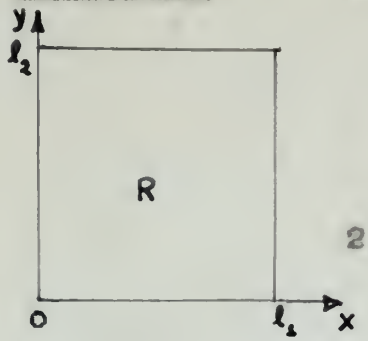
may also be treated by the methods of this chapter.

CHAPTER II

The Characteristic Initial Value Problem for $u_{xy} = f(x,y;u;u_x,u_y)$.

For convenience of reference we first state the following theorem, whose proof is based on the method of successive approximations. The proof of existence was given by É. PICARD [1], while the proof of uniqueness may be found in E. KAMKE [2] p. 410.

Theorem 1. 1) $f(x,y;u;p,q) \in C(B), B: \begin{cases} 0 \le x \le \ell_1 \\ 0 \le y \le \ell_2 \\ -a \le u \le a \\ -b_1 \le p \le b_1 \\ -b_2 \le q \le b_2 \end{cases}$



2) f is Lipschitzian on B; i.e. there exists a positive constant K such that for $(x,y;u_1;p_1,q_1) \in B$, $(x,y;u_2;p_2,q_2) \in B$,

$$|f(x,y;u_1;p_1,q_1)-f(x,y;u_2;p_2,q_2)| \le K\left\{|u_1-u_2| + |p_1-p_2|+|q_1-q_2|\right\}$$

3) $M\ell_1\ell_2 \le a$, $M\ell_1 \le b_2$, $M\ell_2 \le b_1$, where $M = \max|f|$ on B.

$\Longrightarrow$ 4) There exists one and only one function $u(x,y) \in C'(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \le x \le \ell_1 \\ 0 \le y \le \ell_2 \end{cases}$, such that for each $(x,y) \in R$ the point $(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)) \in B$, and $u_{xy}(x,y) = f(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y))$, $u(x,0)) = 0$, $u(0,y) = 0$ for each $(x,y) \in R$.

Remarks. a) Suppose we prescribe $u(x,0) = U(x)$, $u(0,y) = V(y)$ where $U(x) \in C'([0,\ell_1])$, $V(y) \in C'([0,\ell_2])$ and $U(0) = V(0)$. Consider the function $w(x,y) = U(x) + V(y) - U(0)$. Clearly, $w_{xy}(x,y) = 0$ and $w(x,0) = U(x)$, $w(o,y) = V(y)$ hence the function $v = u - w$ must satisfy $v_{xy} = f(x,y;\, v + w;\, v_x + w_x,\, v_y + w_y)$, $v(x,0) = v(0,y) = 0$, a problem of the type covered by Theorem 1.

b) Suppose $f \in C$, bounded and Lipschitzian in the domain B' :
$$\begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \\ -\infty < u < \infty \\ -\infty < p < \infty \\ -\infty < q < \infty \end{cases}$$
Then hypothesis 3) is immediately satisfied.

Following an approach used by M. MULLER [3] p. 632, in dealing with a system of first order ordinary differential equations, we are led to this improvement of the above theorem:

Theorem 1a. 1)

2)' f is partially Lipschitzian on B; i.e. there exists a positive constant K such that for $(x,y;\, u;\, p_1,q_1) \in B$, $(x,y;\, u;\, p_2,q_2) \in B$, $|f(x,y;\, u;\, p_1,q_1) - f(x,y;\, u;\, p_2,q_2)|$ $\leq K\{|p_1-p_2| + |q_1-q_2|\}$.

3)

$\Rightarrow$ 4)' There exists at least one function $u(x,y) \in C'(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$ such that for each $(x,y) \in R$

the point $(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)) \in B$, and $u_{xy}(x,y) = f(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y))$, $u(x,0) = 0$, $u(0,y) = 0$ for each $(x,y) \in R$.

Proof. According to WEIERSTRASS' celebrated theorem [4] p. 1147, on polynomial approximations to a continuous function, there exists a sequence of polynomials, $\{g_\lambda\}(x,y;u;p,q)$, converging uniformly to $f(x,y;\ u;\ p,q)$ on B. We designate this uniform convergence by the notation $\{g_\lambda\} \xrightarrow{\text{unif}} f$ on B.

We extend f and the polynomials g_λ, $(\lambda = 1,2,\cdots)$, over the domain B to the domain B', defined in the remark b) above, by the definition

$$f(x,y;\ u;\ p,q) = f(x,y;\ \bar{u};\ \bar{p},\bar{q})$$

$$g_\lambda(x,y;\ u;\ p,q) = g_\lambda(x,y;\ \bar{u};\ \bar{p},\bar{q}), \quad (\lambda = 1,2,\cdots),$$

(2.1) where

$$\begin{array}{lll} \bar{u} = u \text{ if } -a \leq u \leq a, & \bar{p} = p \text{ if } -b_1 \leq p \leq b_1, & \bar{q} = q \text{ if } -b_2 \leq q \leq b_2. \\ \bar{u} = a \text{ if } a < u & \bar{p} = b_1 \text{ if } b_1 < p & \bar{q} = b_2 \text{ if } b_2 < q \\ \bar{u} = -a \text{ if } u < -a & \bar{p} = -b_1 \text{ if } p < -b_1 & \bar{q} = -b_2 \text{ if } q < -b_2 \end{array}$$

From this extended definition we see that $|f| \leq M$ in B'. Since $\{g_\lambda\} \xrightarrow{\text{unif}} f$ in B', there exists a constant $L > 0$ such that $|g_\lambda| \leq L$ in B' and for all λ. The functions g_λ, $(\lambda = 1,2,\cdots)$ are uniformly continuous in B', moreover they possess bounded difference quotients with respect to the arguments u, p and q everywhere in B'. Hence in B', for each function g_λ there exists a constant $K_\lambda > 0$ such that

$$|g_\lambda(x,y;\,u_1;\,p_1,q_1) - g_\lambda(x,y;\,u_2;\,p_2,q_2)| \le K_\lambda\{|u_1-u_2|+|p_1-p_2| + |q_1-q_2|\}. \tag{2.2}$$

Thus, by Theorem 1, to each g_λ there corresponds one and only one function $u_\lambda(x,y) \in C'(R)$, $u_{\lambda,xy}(x,y) \in C(R)$ satisfying

$$\begin{cases} u_{\lambda,xy} = g_\lambda(x,y;\,u_\lambda(x,y);\,u_{\lambda,x}(x,y),\,u_{\lambda,y}(x,y)), \\ u_\lambda(x,0) = 0, \quad u_\lambda(0,y) = 0 \quad \text{for each } (x,y) \in R. \end{cases} \tag{2.3}$$

We may express the characteristic initial value problem for each u_λ in the form of an equivalent integral equation

$$u_\lambda(x,y) = \int_0^x d\xi \int_0^y g_\lambda(\xi,\eta;\,u_\lambda(\xi,\eta);\,u_{\lambda,x}(\xi,\eta),\, u_{\lambda,y}(\xi,\eta))\,d\eta. \tag{2.4}$$

By differentiation,

$$u_{\lambda,x}(x,y) = \int_0^y g_\lambda(x,\eta;\,u_\lambda(x,\eta);\,u_{\lambda,x}(x,\eta),\,u_{\lambda,y}(x,\eta))\,d\eta \tag{2.5}$$

$$u_{\lambda,y}(x,y) = \int_0^x g_\lambda(\xi,y;\,u_\lambda(\xi,y);\,u_{\lambda,x}(\xi,y),\,u_{\lambda,y}(\xi,y))\,d\xi. \tag{2.6}$$

We now show that the sequences $\{u_\lambda\}$, $\{u_{\lambda,x}\}$, $\{u_{\lambda,y}\}$ are each uniformly bounded and equicontinuous on R. For the sequence $\{u_\lambda\}$ this follows directly from the integral expression (2.4), for, given $x,\ x_1,\ x_2 \in [0,\ell_1]$ and $y,\ y_1,\ y_2 \in [0,\ell_2]$,

$$|u_\lambda(x,y)| \le L\,\ell_1\ell_2, \qquad (\lambda = 1,2,\cdots) \tag{2.7}$$

$$|u_\lambda(x_1,y_1) - u_\lambda(x_2,y_2)| \le L|x_1-x_2|\,|y_1-y_2| + L\,\ell_2\,|x_1-x_2| + L\,\ell_1\,|y_1-y_2|, \quad (\lambda = 1,2,\cdots) \tag{2.8}$$

The uniform boundedness of $\{u_{\lambda,x}\}$ and of $\{u_{\lambda,y}\}$ follow directly from (2.5) and (2.6), respectively, for, given $(x,y) \in R$,

(2.9) $\quad |u_{\lambda,x}(x,y)| \leqslant L\,\ell_2, \qquad (\lambda = 1,2,\cdots)$

(2.10) $\quad |u_{\lambda,y}(x,y)| \leqslant L\,\ell_1, \qquad (\lambda = 1,2,\cdots).$

We base the proof of the equicontinuity of the functions of the sequence $\{u_{\lambda,x}\}$ upon the following two lemmas, the first of which is due to T. H. GRONWALL [5].

Lemma 1. 1) $\quad Z(y) \in C([0,\ell])$

(2.11) 2) $\quad 0 \leqslant Z(y) \leqslant \int_0^y (MZ(\eta) + A)\,d\eta + B \quad$ for $y \in [0,\ell]$
where M, A and B are constants $\geqslant 0$.

(2.12) 3) $\quad 0 \leqslant Z(y) \leqslant (A\ell + B)\,e^{M\ell} \quad$ for $y \in [0,\ell]$.

Lemma 2. Given $\mu > 0$, $\zeta > 0$, there exist δ, a positive constant depending upon μ alone, and N, a positive integer depending upon ζ alone, such that whenever $(x_1,y) \in R$, $(x_2,y) \in R$, $|x_1 - x_2| < \delta$ and $\lambda > N$,

(2.13) $$|u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y)| \leqslant K\int_0^y |u_{\lambda,x}(x_2,\eta) - u_{\lambda,x}(x_1,\eta)|\,d\eta + \mu + \zeta$$

where K is the partial Lipschitz constant for $f(x,y;\,u;\,p,q)$.

Assume, for the moment, the validity of these two lemmas. Each of the functions $u_{\lambda,x}$ is certainly uniformly continuous on R; hence, if we let $Z(y) = |u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y)|$ for any particular $\lambda > N$,

we have immediately that for $|x_2 - x_1| < \delta$,

$$(2.14) \quad |u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y)| \leq (\mu + \zeta)\, e^{K\ell/2}.$$

Suppose $(x_1, y_1) \in R$, $(x_2, y_2) \in R$, then certainly $(x_2, y_1) \in R$ and

$$(2.15) \quad |u_{\lambda,x}(x_2,y_2) - u_{\lambda,x}(x_1,y_1)| \leq |u_{\lambda,x}(x_2,y_2) - u_{\lambda,x}(x_2,y_1)| + |u_{\lambda,x}(x_2,y_1) - u_{\lambda,x}(x_1,y_1)| , \quad (\lambda = 1,2,\cdots).$$

By (2.5) we have that

$$(2.16) \quad |u_{\lambda,x}(x_2,y_2) - u_{\lambda,x}(x_2,y_1)| \leq L\, |y_2 - y_1| , \quad (\lambda = 1,2,\cdots).$$

Inequalities (2.14), (2.15) and (2.16) yield immediately the equicontinuity on R of the functions of the sequence $\{u_{\lambda,x}\}$; for, given $\epsilon > 0$, we first choose $\mu > 0$ and $\zeta > 0$ such that

$$(2.17) \qquad \mu + \zeta < \frac{\epsilon}{2e^{K\ell/2}}$$

and let δ and N be the corresponding constants given by Lemma 2. By the uniform continuity on R of each of the functions $u_{\lambda,x}$, there exists a positive constant δ_N, depending on ϵ alone, such that

$$|x_1 - x_2| < \delta_N \text{ and } |y_1 - y_2| < \delta_N \Rightarrow$$

$$(2.18) \quad |u_{\lambda,x}(x_2,y_2) - u_{\lambda,x}(x_1,y_1)| < \epsilon , \quad (\lambda = 1,2,\cdots, N).$$

Setting $\delta_0 = \min(\delta, \delta_N, \frac{\epsilon}{2L})$ we obtain

$$|x_1-x_2|<\delta_0 \quad \text{and} \quad |y_1-y_2|<\delta_0 \Rightarrow$$

(2.19) $\left|u_{\lambda,x}(x_2,y_2)-u_{\lambda,x}(x_1,y_1)\right|<\epsilon$, for $\lambda = 1,2,\cdots,N,N+1,\cdots$

Proof of Lemma 1: Let $Z(y) = e^{My}\cdot w(y)$, without loss, for we may always choose $w(y) = e^{-My}\cdot Z(y)$. $w(y) \in C([0,\ell])$ and hence attains a maximum thereon. Let w_{max} occur at $y = y_1$, then

$$\begin{aligned} 0 \le e^{My_1}\, w_{max} &\le \int_0^{y_1} (M\, e^{M\eta}\, w(\eta) + A)\, d\eta + B \\ &\le w_{max}\int_0^{y_1} M\, e^{M\eta}\, d\eta + A\, y_1 + B \\ &= w_{max}(e^{My_1}-1) + A\, y_1 + B \end{aligned}$$

Thus $0 \le w_{max} \le A\, y_1 + B \le A\ell + B$ and hence

$$0 \le Z(y) \le (A\ell + B)\, e^{M\ell} \quad \text{for } y \in [0,\ell].$$

Proof of Lemma 2:

$$\begin{aligned} u_{\lambda,x}(x_2,y)-u_{\lambda,x}(x_1,y) &= \int_0^y \big[g_\lambda(x_2,\eta;\, u_\lambda(x_2,\eta);\, u_{\lambda,x}(x_2,\eta),\, u_{\lambda,y}(x_2,\eta)) \\ &\quad - g_\lambda(x_1,\eta;\, u_\lambda(x_1,\eta);\, u_{\lambda,x}(x_1,\eta),\, u_{\lambda,y}(x_1,\eta))\big]\, d\eta \\ &= \int_0^y \big[g_\lambda(x_2,\eta;\, u_\lambda(x_2,\eta);\, u_{\lambda,x}(x_2,\eta),\, u_{\lambda,y}(x_2,\eta)) \\ &\quad - f(x_2,\eta;\, u_\lambda(x_2,\eta);\, u_{\lambda,x}(x_2,\eta),\, u_{\lambda,y}(x_2,\eta))\big]\, d\eta \\ &\quad + \int_0^y \big[f(x_2,\eta;\, u_\lambda(x_2,\eta);\, u_{\lambda,x}(x_2,\eta),\, u_{\lambda,y}(x_2,\eta)) \end{aligned} \tag{2.20}$$

(2.20)
(Continued)

$$- f(x_2,\eta;\ u_\lambda(x_2,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_2,\eta)]\, d\eta$$

$$+\int_0^y [f(x_2,\eta;\ u_\lambda(x_2,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_2,\eta))$$

$$- f(x_1,\eta;\ u_\lambda(x_1,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_1,\eta))]\, d\eta$$

$$+\int_0^y [f(x_1,\eta;\ u_\lambda(x_1,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_1,\eta))$$

$$- g_\lambda(x_1,\eta;\ u_\lambda(x_1,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_1,\eta))]\, d\eta$$

$$(\lambda = 1,2,\cdots).$$

Since $\{g_\lambda\} \xrightarrow{\text{unif}} f$ on R', given $\zeta > 0$, there exists a positive integer N, depending upon ζ alone, such that for $\lambda > N$,

(2.21) $$\left|\int_0^y [g_\lambda(x_2,\eta;\ u_\lambda(x_2,\eta);\ u_{\lambda,x}(x_2,\eta),\ u_{\lambda,y}(x_2,\eta)) - f(x_2,\eta;\ u_\lambda(x_2,\eta);\ u_{\lambda,x}(x_2,\eta),\ u_{\lambda,y}(x_2,\eta))]\, d\eta\right|$$
$$+\left|\int_0^y [f(x_1,\eta;\ u_\lambda(x_1,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_1,\eta)) - g_\lambda(x_1,\eta;\ u_\lambda(x_1,\eta);\ u_{\lambda,x}(x_1,\eta),\ u_{\lambda,y}(x_1,\eta))]\, d\eta\right| < \zeta$$

By hypothesis 2)',

(2.22) $$\left|\int_0^y [f(x_2,\eta;\ u_\lambda(x_2,\eta);\ u_{\lambda,x}(x_2,\eta),\ u_{\lambda,y}(x_2,\eta)) -\right.$$

(2.22)
(Continued) $-f(x_2, \eta; u_\lambda(x_2,\eta); u_{\lambda,x}(x_1,\eta), u_{\lambda,y}(x_2,\eta))] \, d\eta|$

$$\leq K \int_0^y |u_{\lambda,x}(x_2,\eta) - u_{\lambda,x}(x_1,\eta)| \, d\eta \ , \quad (\lambda = 1,2,\cdots)$$

Since f is uniformly continuous on R', the functions of the sequence $\{u_\lambda\}$ are equicontinuous on R, and $|u_{\lambda,y}(x_2,\eta) - u_{\lambda,y}(x_1,\eta)| \leq L \, |x_2 - x_1|$, $(\lambda = 1,2,\cdots)$, it follows that given $\mu > 0$ there exists a positive constant δ , depending upon μ alone, such that for $|x_2 - x_1| < \delta$,

(2.23) $$\left| \int_0^y [f(x_2,\eta; u_\lambda(x_2,\eta); u_{\lambda,x}(x_1,\eta), u_{\lambda,y}(x_2,\eta)) - f(x_1,\eta; u_\lambda(x_1,\eta); u_{\lambda,x}(x_1,\eta), u_{\lambda,y}(x_1,\eta))] \, d\eta \right| < \mu,$$

$(\lambda = 1,2,\cdots)$.

Therefore, from (2.21), (2.22) and (2.23), by virtue of (2.20) we obtain that for $\lambda > N$ and $|x_2 - x_1| < \delta$,

(2.13) $$|u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y)| < K \int_0^y |u_{\lambda,x}(x_2,\eta) - u_{\lambda,x}(x_1,\eta)| \, d\eta + \mu + \zeta$$

thus verifying Lemma 2.

The proof of the equicontinuity of the functions of the sequence $\{u_{\lambda,y}\}$ follows precisely the same steps as that for the sequence $\{u_{\lambda,x}\}$.

We now invoke the well-known theorem of C. ARZELA [3] p. 1144:

"Given a set F of functions f defined and continuous on the closed bounded set A, then the necessary and sufficient conditions that each sequence of functions contained in F possesses

a subsequence uniformly convergent on A are that F be uniformly bounded and equicontinuous."

By Theorem 1, there exists a unique triple $(u_\lambda; u_{\lambda,x}; u_{\lambda,y})$ corresponding to g_λ for each λ. Since any subsequence of a uniformly convergent sequence is likewise uniformly convergent; and, since any subsequence of a uniformly bounded and equicontinuous sequence is likewise uniformly bounded and equicontinuous; there exists a subsequence $\{g^*_\lambda\}$ of the sequence $\{g_\lambda\}$ such that the corresponding sequences

$$(2.24)\quad \{u^*_\lambda\} \xrightarrow{\text{unif}} u, \quad \{u^*_{\lambda,x}\} \xrightarrow{\text{unif}} v, \quad \{u^*_{\lambda,y}\} \xrightarrow{\text{unif}} w,$$

where $u, v, w \in C(R)$. This results from the following successive extractions of subsequences:

$\{u_\lambda\}$ is equicontinuous and uniformly bounded on R, hence there exists a subsequence $\{u^1_\lambda\}$ of $\{u_\lambda\}$ uniformly convergent on R. $\{u^1_{\lambda,x}\}$ is equicontinuous and uniformly bounded on R, hence there exists a subsequence $\{u^2_{\lambda,x}\}$ of $\{u^1_{\lambda,x}\}$ uniformly convergent on R. $\{u^2_{\lambda,y}\}$ is equicontinuous and uniformly bounded on R, hence there exists a subsequence $\{u^*_{\lambda,y}\}$ of $\{u^2_{\lambda,y}\}$ uniformly convergent on R. But, by the one-to-one correspondence mentioned above, $\{u^*_{\lambda,x}\}$ is a subsequence of $\{u^2_{\lambda,x}\}$ while $\{u^*_\lambda\}$ is a subsequence of $\{u^1_\lambda\}$. Thus $\{u^*_{\lambda,x}\}$ and $\{u^*_\lambda\}$ are each uniformly convergent on R.

Writing, with the notation $u^*_0 = u^*_{0,x} = u^*_{0,y} = 0$,

$$(2.25)\quad u^*_\lambda = \sum_{k=1}^{\lambda}(u^*_k - u^*_{k-1}),\ u^*_{\lambda,x} = \sum_{k=1}^{\lambda}(u^*_{k,x} - u^*_{k-1,x}),$$

$$u^*_{\lambda,y} = \sum_{k=1}^{\lambda}(u^*_{k,y} - u^*_{k-1,y}),\quad (\lambda = 1,2,\cdots),$$

we see that the conditions for differentiation under the summation sign for infinite series are satisfied by (2.24) and the fact that $u^*_\lambda \in C'(R)$, $(\lambda = 1,2,\cdots)$. Hence

$$(2.26)\quad v(x,y) = u_x(x,y),\quad w(x,y) = u_y(x,y)\quad \text{for } (x,y) \in R$$

We show that the function u so determined satisfies the integral equation equivalent to the original characteristic initial value problem, i.e.

$$(2.27)\quad u(x,y) = \int_0^x d\xi \int_0^y f(\xi,\eta;\ u(\xi,\eta);\ u_x(\xi,\eta),\ u_y(\xi,\eta))d\eta$$

for $(x,y) \in R$.

For any λ, by (2.4),

$$(2.28)\quad \left| u(x,y) - \int_0^x d\xi \int_0^y f(\xi,\eta;\ u(\xi,\eta);\ u_x(\xi,\eta),\ u_y(\xi,\eta))\, d\eta \right|$$
$$\leq |u(x,y) - u^*_\lambda(x,y)| + \int_0^x d\xi \int_0^y \left| f(\xi,\eta;\ u(\xi,\eta);\ u_x(\xi,\eta),\right.$$
$$\left. u_y(\xi,\eta)) - f(\xi,\eta;\ u^*_\lambda(\xi,\eta);\ u^*_{\lambda,x}(\xi,\eta),\ u^*_{\lambda,y}(\xi,\eta))\right| d\eta$$
$$+ \int_0^x d\xi \int_0^y \left| f(\xi,\eta;\ u^*_\lambda(\xi,\eta);\ u^*_{\lambda,x}(\xi,\eta),\ u^*_{\lambda,y}(\xi,\eta))\right.$$
$$\left. - g^*_\lambda(\xi,\eta;\ u^*_\lambda(\xi,\eta);\ u^*_{\lambda,x}(\xi,\eta),\ u^*_{\lambda,y}(\xi,\eta))\right|\ d\eta$$

Since $\{g^*_\lambda\} \xrightarrow{\text{unif}} f$ on R', $\{u^*_\lambda\} \xrightarrow{\text{unif}} u$ on R, given $\epsilon > 0$ and $(x,y) \in R$, there exists a positive integer N_1, depending upon ϵ alone, such that for $\lambda > N_1$,

(2.29) $$|u(x,y) - u^*_\lambda(x,y)| < \epsilon \quad ,$$

(2.30) $$\int_0^x d\xi \int_0^y |f(\xi,\eta; u^*_\lambda(\xi,\eta); u^*_{\lambda,x}(\xi,\eta), u^*_{\lambda,y}(\xi,\eta))$$
$$- g^*_\lambda(\xi,\eta; u^*_\lambda(\xi,\eta); u^*_{\lambda,x}(\xi,\eta), u^*_{\lambda,y}(\xi,\eta))| \; d\eta$$
$$< \epsilon \ell_1 \ell_2 \, .$$

Moreover, since f is uniformly continuous in B' while $\{u^*_\lambda\}$, $\{u^*_{\lambda,x}\}$, $\{u^*_{\lambda,y}\}$ converge uniformly on R to u, u_x, u_y respectively, there exists a positive integer N_2, depending on ϵ alone, such that for $\lambda > N_2$,

(2.31) $$\int_0^x d\xi \int_0^y |f(\xi,\eta; u(\xi,\eta); u_x(\xi,\eta), u_y(\xi,\eta))$$
$$- f(\xi,\eta; u^*_\lambda(\xi,\eta); u^*_{\lambda,x}(\xi,\eta), u^*_{\lambda,y}(\xi,\eta))| \; d\eta$$
$$< \epsilon \ell_1 \ell_2 \, .$$

Introducing (2.29), (2.30) and (2.31) into (2.28), we obtain that for $\lambda > \max(N_1, N_2)$

(2.32) $$|u(x,y) - \int_0^x d\xi \int_0^y f(\xi,\eta; u(\xi,\eta); u_x(\xi,\eta), u_y(\xi,\eta))$$
$$< \epsilon(1 + 2\ell_1 \ell_2)$$

But ϵ is arbitrary, hence (2.27) is verified for each $(x,y) \in R$

We must verify the one additional fact that for each $(x,y) \in R$, $(x,y; u(x,y); u_x(x,y), u_y(x,y)) \in B$, instead of just belonging to B'.

By differentiation from (2.27),

$$(2.33)\quad u_x(x,y) = \int_0^y f(x,\eta;\ u(x,\eta);\ u_x(x,\eta),\ u_y(x,\eta))\, d\eta$$

$$(2.34)\quad u_y(x,y) = \int_0^x f(\xi,y;\ u(\xi,y);\ u_x(\xi,y),\ u_y(\xi,y))\, d\xi .$$

Hence, from the extended definition of f, (2.1), and hypothesis 3),

$$(2.35)\quad |u(x,y)| \leq \int_0^x d\xi \int_0^y |f(\xi,\eta;\ u(\xi,\eta);\ u_x(\xi,\eta),\ u_y(\xi,\eta))|\, d\eta$$

$$\leq M\ell_1\ell_2 \leq a$$

$$(2.36)\quad |u_x(x,y)| \leq \int_0^y |f(x,\eta;\ u(x,\eta);\ u_x(x,\eta),\ u_y(x,\eta))|\, d\eta$$

$$\leq M\ell_2 \leq b_1$$

$$(2.37)\quad |u_y(x,y)| \leq \int_0^x |f(\xi,y;\ u(\xi,y);\ u_x(\xi,y),\ u_y(\xi,y))|\, d\xi$$

$$\leq M\ell_1 \leq b_2 ,$$

thus completing the proof of Theorem 1a.

Remarks a) and b) to Theorem 1 apply as well to Theorem 1a.

By the following example we show that the integral surfaces for Theorem 1a are not necessarily unique:

Example 1 Consider the characteristic initial value problem:

$$(2.38)\qquad u_{xy} = |u|^{\frac{1}{2}} ;\quad u(x,0) = u(0,y) = 0.$$

Here $f(x,y;\ u;\ p,q) = |u|^{\frac{1}{2}}$ is continuous for all u but fails to satisfy a Lipschitz condition on u at u = 0. Theorem 1a applies

to insure existence of a solution in a sufficiently small neighborhood of the origin. However, at least two solutions, valid for all (x,y) in the finite plane, are directly available. First, $u \equiv 0$ obviously satisfies. Second, if we seek a solution u satisfying

i) $u \geqslant 0$,

ii) there exist functions X, Y such that
$$u(x,y) = X(x) \cdot Y(y);$$

that is, by the method of separation of variables, we obtain immediately the solution $u(x,y) = \frac{1}{16} x^2 y^2$.

For purposes of illustrating the various situations that might occur, we give the following:

Example 2. Consider the characteristic initial value problem:

$$(2.39) \qquad u_{xy} = |u_x|^{\frac{1}{2}} \;;\; u(x,0) = u(0,y) \;;\; 0.$$

Here $f(x,y;\, u;\, p,q) = |p|^{\frac{1}{2}}$ is continuous for all p but fails to satisfy a Lipschitz condition on p at $p = 0$. Since $p(x,0) = u_x(x,0) = 0$ neither Theorem 1 nor Theorem 1a will insure existence of a solution over any domain including a portion of the x axis. Such solutions do exist, however. One is $u \equiv 0$. Under the assumption $p = u_x \geqslant 0$ we obtain another, for now

$$p_y = p^{\frac{1}{2}} \quad \text{or}$$

$$\frac{dp}{p^{\frac{1}{2}}} = 2p^{\frac{1}{2}} = y + c_1.$$

Since $p(x,0) = 0$, $c_1 = 0$ and

$$p = u_x = \frac{y^2}{4} \qquad \text{or, integrating,}$$

$$u = \frac{xy^2}{4} + c_2.$$

Since $u(0,y) = 0$, $c_2 = 0$; and hence

$$u = \frac{xy^2}{4}$$

is a second solution valid throughout the finite plane.

In Example 2 consider the function

$$u_0(x,y) = \begin{cases} 0 & \text{for} \quad x \leq 0 \\ \frac{xy^2}{4} & \text{for} \quad x \geqslant 0 \,. \end{cases}$$

u_0 is continuous for all (x,y) and satisfies the initial value problem (2.39) everywhere except along the y axis, where for $y \neq 0$, $u_{0x}(0,y)$ does not exist. Roughly speaking, u_0 is a continuous integral surface of problem (2.39) having a jump in the normal first derivative across a characteristic.

For equations meeting the continuity, boundedness and partial Lipschitz requirements of Theorem 1a we cannot match integral surfaces in the above fashion to obtain first derivative jumps across characteristics. This follows from the fact that if we prescribe $u(a,y) = V(y) \in C'([0,\ell_2])$ along the characteristic $x=a$, $a \in [0,\ell_1]$, then

$$(2.40) \quad \begin{cases} p_y(a,y) = f(a,y; V(y); p(a,y), V'(y)) \\ p(a,0) = 0 \end{cases}$$

represents a first order ordinary differential equation for the

unknown function $p = u_x$ under a one point boundary condition. The conditions that f be continuous, bounded and partially Lipschitzian are sufficient to insure the existence of a unique determination of $u_x(a,y)$ for $y \in [0,\ell_2]$. Note that in Example 2 the function f was continuous only and hence the determination of u_x from the above ordinary differential equation was not unique, thus admitting the possibility of a jump in u_x. The conditions for the determination of u_y along a characteristic $y =$ const. are similar.

The above remarks serve to permit the extension of the domain of existence of the integral surfaces of Theorems 1 and 1a from R to R^*: $\begin{cases} -\ell_1 \le x \le \ell_1 \\ -\ell_2 \le y \le \ell_2 \end{cases}$. The arguments for the existence may be made applicable to other quadrants than the first by mere coordinate reflections. Moreover the integrals obtained in the separate quadrants must have first order contacts with each other along the coordinate axes by the above reasoning from ordinary differential equation theory. Hence we may obtain existence and uniqueness over the domain R^* by replacing B by B^*:

$$B^*: \begin{cases} -\ell_1 \le x \le \ell_1 \\ -\ell_2 \le y \le \ell_2 \\ -a \le u \le a \\ -b_1 \le p \le b_1 \\ -b_2 \le q \le b_2 \end{cases}$$

in Theorem 1; and we obtain simply existence over R^* by replacing B by B^* in Theorem 1a.

In the classical existence theorem for the ordinary differential equation $\frac{dy}{dx} = f(x,y)$, with $y(0) = 0$, the conditions that f

be continuous on $C: \begin{cases} 0 \leq x \leq a \\ -b \leq y \leq b \end{cases}$, with $M = \max |f|$ on C, were shown to be sufficient to insure existence of at least one integral curve $y = y(x)$ for $x \in [0,\alpha]$ with $\alpha \leq \min(a,\frac{b}{M})$. This bound, $\alpha \leq \min(a,\frac{b}{M})$, was shown by A. WINTNER [15] to be a maximal bound in a certain sense. We apply his method to Theorem 1a in the proof of the following:

Theorem 2.

If, in Theorem 1a, we replace B by $B'': \begin{cases} 0 \leq x \leq \ell_1' \\ 0 \leq y \leq \ell_2' \\ -\infty < u < \infty \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{cases}$

and require that f be bounded thereon, then hypothesis 3) in that theorem reduces to

$$3)' \quad \ell_1 \leq \min(\ell_1', \frac{b_2}{M}), \ \ell_2 \leq \min(\ell_2', \frac{b_1}{M}),$$

where $M = \max |f|$ on B''. Moreover, the bounds established by 3)' are maximal bounds in a sense to be explained below.

Proof.

The condition $M \ell_1 \ell_2 \leq a$ of hypothesis 3) is immediately satisfied since a approaches $+\infty$. The conditions $M \ell_1 \leq b_2$, $M \ell_2 \leq b_1$ may be rewritten as in 3)' and are now the only restrictions on ℓ_1 and ℓ_2.

If $\ell_1' \leq \frac{b_2}{M}$, $(\ell_2' \leq \frac{b_1}{M})$, then the conclusion is immediate. For, we may take $f(x,y;\, u;\, p,q) = h(x)$, $(g(y))$, which function is not even defined for $x > \ell_1 = \ell_1'$, $(y > \ell_2 = \ell_2')$.

Suppose $\ell_2' > \frac{b_1}{M}$. Then we consider the sequence of problems:

$$(2.41) \qquad u_{xy} = (2^{1/m} - u_x)^{1/m+1}, \quad u(x,0) = u(o,y) = 0, \quad (m=1,2,\cdots).$$

Setting $p = u_x$, (2.41) becomes

$$p_y(x,y) = (2^{1/m} - p(x,y))^{1/m+1}, \quad p(x,0) = 0.$$

Integrating this ordinary differential equation for p as a function of y, we obtain

$$p(x,y) = 2^{1/m} - \left[2^{1/m+1} - \frac{m}{m+1}\, y\right]^{m+1/m}.$$

But, since $p = u_x$ and $u(0,y) = 0$ we may integrate again to obtain

$$(2.42) \qquad u(x,y) = x\left\{2^{1/m} - \left[\frac{m}{m+1}(C_m - y)\right]^{m+1/m}\right\}$$

where

$$(2.43) \qquad C_m = \frac{m+1}{m}\, 2^{\frac{1}{m+1}}.$$

The line $y = C_m$ is a branch line of the solution u. Under the supposition $\ell_2' > \frac{b_1}{M}$, the desired statement is that $\frac{b_1}{M}$ is a maximal bound on ℓ_2, i.e., for each $\epsilon > 0$, there exists a function $f(x,y;\, u;\, p,q)$, depending on ϵ and satisfying hypotheses 1), 2)' and 3)' on R", such that an integral $u(x,y)$ of the problem corresponding to f has a singularity for some $y \in (\frac{b_1}{M}, \frac{b_1}{M} + \epsilon)$.

Defining

$$f_m(x,y;\ u;\ p,q) = (2^{1/m} - p)^{1/m+1} \text{ for } -2^{1/m+1} \leq p \leq 2^{1/m+1},$$

$(m = 1,2,\cdots)$, we obtain

$b_{1m} = 2^{1/m+1}$, $M_m = (2^{1/m} + 2^{1/m+1})^{1/m+1}$; and, since

$$(2^{1/m} + 2^{1/m+1}) > 2,\ (m = 1,2,\cdots),$$

$$\lim_{m \to \infty} \frac{b_{1m}}{M_m} = 1 - .$$

Moreover, by (2.43),

$$\lim_{m \to \infty} C_m = 1 \quad .$$

Hence, given $\epsilon > 0$, there exists a positive integer M, depending on ϵ alone, such that $m > M \Longrightarrow$

$$\frac{b_{1m}}{M_m} + \epsilon \ > \ C_m > \frac{b_{1m}}{M_m} \quad .$$

Consequently $\dfrac{b_1}{M}$ is a maximal bound on ℓ_2.

To determine that the condition $\ell_1 \leqslant \min\,(\ell_1', \dfrac{b_2}{M})$ is also a maximal bound we consider the sequence of problems.

(2.44) $\quad u_{xy} = (2^{1/m} - u_y)^{1/m+1}$, $\ u(x,0) = u(0,y)$, $(m = 1,2,\cdots)$,

and follow the same line of reasoning as before. Thus Theorem 2 is verified.

The close parallelism between our conclusions and the classical theorems for first order ordinary differential equations

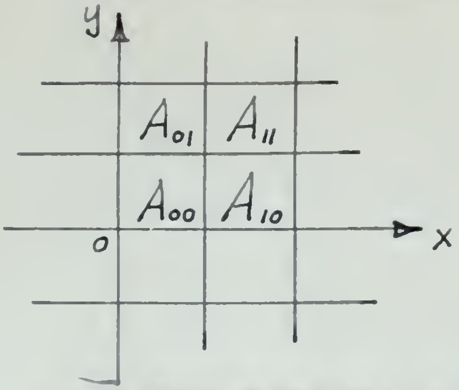
(See F. KAMKE [2]) leads one to suspect that an existence theorem might be proved wherein mere continuity of the function f was demanded. The analogue to the Cauchy polygon method is the attack suggested by the parallelism, and it leads to an existence theorem for the characteristic initial value problem:

$$(2.45) \qquad u_{xy} = f(x,y;\, u) \quad , \quad u(x,0) = u(0,y) = 0.$$

We do not give the proof here; first, because the theorem is a special case of Theorem 1a; and, second, because the steps in the proof are practically identical with those of the Cauchy polygon method for ordinary differential equations.

When $f = f(x,y;\, u;\, p,q)$ and f is merely continuous this attack involves difficulties which we have not been able to resolve. We sketch the method to indicate the source of trouble:

In a neighborhood of the origin a partition π by characteristics is specified where the subregions A_{ij} in the first quadrant are defined as



$$A_{ij}: \begin{cases} x_i \leqq x < x_{i\ 1} \\ y_j \leqq y < y_{j\ 1} \end{cases} \quad (i,j=0,1,2\cdots)$$

We formulate the approximate integral surface u corresponding to the partition π as follows:

$$(2.46) \qquad u_\pi(x,y) = \int_0^x d\xi \int_0^y F_\pi(\xi,\eta)\, d\eta$$

where

$$(2.47)\qquad F_{\pi}(x,y) = f(x_i,\ y_j;\ u_{\pi}(x_i,y_j);\ u_{\pi_x}(x_i,y_j),\ u_{\pi_y}(x_i y_j))$$

for $(x,y) \in A_{ij}$.

The principal difficulty lies in the fact that the derivatives

$$(2.48)\qquad u_{\pi x} = \int_0^y F_{\pi}(x,\eta)\, d\eta \quad \text{and}$$

$$(2.49)\qquad u_{\pi y} = \int_0^x F_{\pi}(\xi,y)\, d\xi$$

are discontinuous across the partition lines $x =$ constant and $y =$ constant, respectively, thus preventing the direct application of ARZELA's theorem on equicontinuous functions when we consider the sequence of approximate integral surfaces formed by partition refinement.

The equation of (2.45) appears to be more amenable than the more general equation involving the first derivatives p and q. G. FUBINI [16] p. 622, by demanding only that $f(x,y;u)$ be continuous and Lipschitzian with respect to u, has proved the existence of a unique integral of $u_{xy} = f(x,y;u)$ satisfying Dirichlet conditions, i.e. the value of u prescribed on a closed contour. This result, while remarkable, is not contradictory since u is shown to have a discontinuity of the second type at one point of the boundary.

We conclude this chapter with the statement of the extension of Theorems 1 and 1a to a system of equations

(2.50) $s_i = f_i(x,y;\ u_1,\cdots,u_n;\ p_1,\cdots p_n, q_1,\cdots,q_n), (i=1,2,\cdots,n)$

satisfying the initial conditions

(2.51) $u_i(x,0) = u_i(0,y) = 0\ , \quad (i=1,2,\cdots,n).$

Theorem 3, below, is a natural extension of Theorem 1. In principle, it was first obtained by O. NICCOLETTI [14] p.7. His statement, however, is not explicit about the bounds on the domain of existence. Moreover, to prove uniqueness he requires the f_i to be of class C'. We obtain the improved statement, Theorem 3, by modifying the arguments of E. KAMKE [2] p. 402 and p. 403 to apply them to the system (2.50).

Theorem 3)

1) $f_i(x,y;\ u_j;\ p_j,q_j)$[2] $\in C(B''),\ B'': \begin{cases} 0 \le x \le \ell_1 \\ 0 \le y \le \ell_2 \\ -a \le u_i \le a \\ -b_1 \le p_i \le b_1 \\ -b_2 \le q_i \le b_2 \end{cases}$

2) The f_i are Lipschitzian on B''; i.e. there exists a positive constant K such that for $(x,y;\ u^1_j;\ p^1_j,\ q^1_j) \in B''$, $(x,y;\ u^2_j;\ p^2_j, q^2_j) \in B''$, and $i = 1,2,\cdots,n,$

$$\left| f_i(x,y;\ u^1_j;\ p^1_j,q^1_j) - f_i(x,y;\ u^2_j;\ p^2_j,q^2_j) \right|$$
$$\le K \sum_{j=1}^{n} \left\{ |u^1_j - u^2_j| + |p^1_j - p^2_j| + |q^1_j - q^2_j| \right\} .$$

3) $M\,\ell_1\,\ell_2 \le a,\ M\,\ell_1 \le b_2,\ M\,\ell_2 \le b_1$ where

$M = \max \left\{ |f_1|, \cdots, |f_n| \right\}$ on B''.

---

[2] Notation: $(x,y;\ u_j;\ p_j,q_j) = (x,y;\ u_1,\cdots,u_n; p_1,\cdots,p_n, q_1,\cdots,q_n).$

$\Longrightarrow$ 4) There exists one and only one set of functions $\{u_1,\cdots,u_n\}$, $u_j(x,y) \in C'(R)$, $u_{j,xy}(x,y) \in C(R)$, $(j=1,\cdots,n)$, where $R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$, such that for each $(x,y) \in R$ the point $(x,y;u_j(x,y);\ u_{j,x}(x,y),\ u_{j,y}(x,y)) \in B''$, and

$$u_{i,xy}(x,y) = f_i(x,y;u_j(x,y);\ u_{j,x}(x,y),\ u_{j,y}(x,y)),$$

$$u_i(x,0) = u_i(0,y) = 0, \quad (i = 1,\cdots,n), \text{ for each } (x,y) \in R.$$

By relaxing hypothesis 2) we obtain the improved theorem below; which stands in the same relation to Theorem 3 that Theorem 1a does to Theorem 1.

Theorem 3a

1)

2)' The f_i are partially Lipschitzian on B''; i.e. there exists a positive constant K such that for $(x,y;\ u_j;\ p_j^1,\ q_j^1) \in B''$, $(x,y;u_j;p^2_j,q^2_j) \in B''$, and $i = 1, 2,\cdots, n$,

$$\left| f_i(x,y;\ u_j;\ p^1_j,\ q^1_j) - f_i(x,y;\ u_j;\ p^2_j,\ q^2_j) \right| \leq K \sum_{j=1}^{n} \left\{ |p^1_j - p^2_j| + |q^1_j - q^2_j| \right\}.$$

3)

$\Longrightarrow$ 4)' There exists at least one set of functions $\{u_1,\cdots,u_n\}$, $u_j(x,y) \in C'(R)$, $u_{j,xy}(x,y) \in C(R)$, $(j=1,\cdots,n)$, where

$R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$, such that for each $(x,y) \in R$ the point

$(x,y;\ u_j(x,y);\ u_{j,x}(x,y),\ u_{j,y}(x,y)) \in B''$, and

$u_{i,xy}(x,y) = f_i(x,y;\ u_j(x,y);\ u_{j,x}(x,y),\ u_{j,y}(x,y))$,

$u_i(x,0) = u_i(0,y) = 0$, $(i = 1,\cdots n)$, for each $(x,y) \in R$.

The proof of Theorem 3a is essentially a step by step repetition of that for Theorem 1a. WEIERSTRASS' theorem tells us that for each positive integer i there exists a sequence of polynomials $\{g_{i\lambda}\}$ $(x,y;\ u_j;\ p_j,q_j)$, $(\lambda = 1,2,\cdots)$, converging uniformly on B'' to $f_i(x,y;\ u_j;\ p_j,q_j)$. We extend the $g_{i\lambda}$ and the f_i as before and obtain that there exist positive constants L_i such that for each i $|g_{i\lambda}| \leq L_i$ on B'', extended, and for all λ. We let $L = \max\{L_1,\cdots, L_n\}$ and proceed as before, using Theorem 3 in place of Theorem 1 to obtain the integral $u_{i\lambda}$ associated with each $g_{i\lambda}$.

We note only one point of significant difference in the arguments. In place of inequality (2.13) of Lemma 2 we now have the inequalities

$$|u_{i\lambda,x}(x_2,y) - u_{i\lambda,x}(x_1,y)|$$
$$\leq K\int_0^y \left\{\sum_{j=1}^{n} |u_{j\lambda,x}(x_2,\eta) - u_{j\lambda,x}(x_1,\eta)|\right\} d\eta$$
$$(i = 1,\cdots,n).$$

Summing these, and letting

$$Z(y) = \sum_{i=1}^{n} |u_{i\lambda,x}(x_2,y) - u_{i\lambda,x}(x_1,y)|,$$

we obtain

$$0 \leq Z(y) \leq Kn \int_0^y Z(\eta)\, d\eta + n(\mu + \zeta)$$

to which Lemma 1 applies. Thus the equicontinuity of each of the sequences $\{u_{i\lambda,x}\}$, $(i = 1,\cdots,n)$ is assured.

Remarks a) and b) to Theorems 1 and 1a apply, with obvious modifications, to Theorems 3 and 3a. Moreover, as before, we may extend the domain of existence of the integral surfaces of Theorems 3 and 3a from R to R*: $\begin{cases} -\ell_1 \leq x \leq \ell_1 \\ -\ell_2 \leq y \leq \ell_2 \end{cases}$.

The set of functions $\{u_1,\cdots, u_n\}$ representing the solution to the problem of Theorem 3a cannot be shown to be unique. This is made evident by extending Example 1 to the system

$$\begin{array}{lll} u_{1,xy} = |u_1|^{\frac{1}{2}} & , & u_1(x,0) = u_1(0,y) = 0 \\ u_{2,xy} = 0 & , & u_2(x,0) = u_2(0,y) = 0 \\ \vdots & & \vdots \qquad\qquad \vdots \\ u_{n,xy} = 0 & , & u_n(x,0) = u_n(0,y) = 0 \end{array}$$

for which $u_i \equiv 0$ $\qquad (i = 2,\cdots,n)$
while $u_1 \equiv 0$ or $u_1 = \frac{1}{16} x^2 y^2$. Thus at least two sets of solutions are possible for this system which satisfies the hypotheses of Theorem 3a.

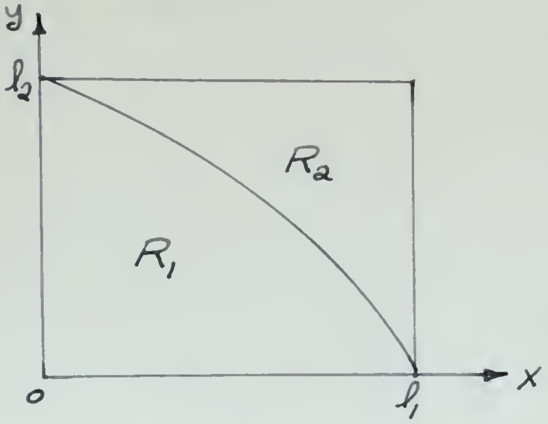
CHAPTER III

The Cauchy Problem for $u_{xy} = f(x,y;\ u;\ u_x, u_y)$.

The development of this chapter closely parallels that of Chapter II. Consequently, the notation will be abridged, in particular with respect to the arguments of functions; and the proofs will be merely outlined to show minor variations from the statements in Chapter 2.

For reference, we state the following theorem proved first for systems of equations by O: NICCOLETTI [14] p. 7. Our statement may be easily inferred from that of E. KAMKE [2] p. 405 and p. 410, by a slight modification of his proof.

Theorem 4



1) $f(x,y;u;p,q) \in C(B)$,

$$B: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \\ -a \leq u \leq a \\ -b_1 \leq p \leq b_1 \\ -b_2 \leq q \leq b_2 \end{cases}$$

2) f is Lipschitzian on B, (as defined in Theorem 1).

3) $M \ell_1 \ell_2 \leq a$, $M \ell_1 \leq b_2$, $M \ell_2 \leq b_1$, where $M = \max |f|$ on B

4) $\gamma: \begin{cases} 0 \leq x \leq \ell_1 \\ y = \varphi(x) \end{cases}$ where $\varphi(x) \in C'([0,\ell_1])$, $\varphi'(x) \neq 0$ for $x \in [0,\ell_1]$ and $\varphi(0) = \ell_2$, $\varphi(\ell_1) = 0$.

$\Longrightarrow$ 5) There exists one and only one function $u(x,y) \in C'(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$, such that for each $(x,y) \in R$, the point $(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)) \in B$, and

$$u_{xy}(x,y) = f(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)),$$

$$u(x, \varphi(x)) = u_x(x, \varphi(x)) = u_y(x, \varphi(x)) = 0$$

for each $(x,y) \in R$.

Remarks c) Suppose we prescribe $u(x, \varphi(x)) = U(x)$, $u_x(x, \varphi(x)) = P(x)$, $u_y(x, \varphi(x)) = Q(x)$ where $U(x) \in C'([0,\ell_1])$ while $P(x), Q(x) \in C([0,\ell_1])$. Our prescription must satisfy the strip condition $U' = P + Q \cdot \varphi'$ for each $x \in [0,\ell_1]$. Consider the function $w(x,y) = U(x) + (y - \varphi(x))\, Q(x)$. Clearly, $w_{xy} = Q'(x)$ while $w(x, \varphi(x)) = U(x)$, $w_x(x, \varphi(x)) = P(x)$, and $w_y(x, \varphi(x)) = Q(x)$. Hence the function $v = u - w$ must satisfy $v_{xy} = Q'(x) + f(x,y;\ v + w;\ v_x + w_x,\ v_y + w_y)$, with $v(x, \varphi(x)) = v_x(x, \varphi(x)) = v_y(x, \varphi(x)) = 0$, a problem of the type covered by Theorem 4.

d) Hypothesis 4) of Theorem 4 is more restrictive than it need be. At isolated points of Υ we may have a horizontal or vertical tangent, provided that Υ does not cross the same characteristic more than once. For, under these conditions the inverse function ψ to φ will exist and be continuous for all $y \in [0,\ell_2]$.

Our improvement of this theorem is as follows:

Theorem 4a

1)

2)' f is partially Lipschitzian on D, (as defined in Theorem 1a).

3)

4)

$\Rightarrow$ 5) There exists at least one function $u(x,y) \in C'(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$, such that for each $(x,y) \in R$, the point $(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)) \in D$, and

$$u_{xy}(x,y) = f(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)),$$

$$u(x, \varphi(x)) = u_x(x, \varphi(x)) = u_y(x, \varphi(x)) = 0$$

for each $(x,y) \in R$.

Outline of proof.

The path Υ may also be expressed as $\Upsilon: \begin{cases} x = \psi(y) \\ 0 \leq y \leq \ell_2 \end{cases}$ where $\psi(y) \in C'([0,\ell_2])$, $\psi'(y) \nleq 0$ for $y \in [0,\ell_2]$. ψ is the inverse function to φ.

We may express the problem as the integral equation

$$(3.1) \qquad u(x,y) = \int_{\psi(y)}^{x} d\xi \int_{\varphi(\xi)}^{y} f(\xi, \eta;u;u_x, u_y)\, d\eta = \int_{\varphi(x)}^{y} d\eta \int_{\psi(\eta)}^{x} f(\xi, \eta;u;u_x,u_y)\, d\xi$$

Whence

$$(3.2) \qquad u_x(x,y) = \int_{\varphi(x)}^{y} f(x, \eta;u;u_x,u_y)\, d\eta$$

$$(3.3) \qquad u_y(x,y) = \int_{\psi(y)}^{x} f(\xi, y;\ u;\ u_x,u_y)\, d\xi .$$

By WEIERSTRASS' theorem, there exists a sequence of polynomials $\{g_\lambda\} \xrightarrow{\text{unif.}} f$ on B. We extend the domain of definition of f and the polynomials g_λ over B to B' by definition (2.1).

We obtain again the constant $L > 0$ such that $|g_\lambda| \leq L$ in B' for all λ. Moreover, for each g_λ the Lipschitz condition (2.2) is satisfied. Thus, by Theorem 4, for each λ there exists a unique solution u_λ to the problem

$$(3.4) \quad \begin{cases} u_{\lambda,xy} = g_\lambda(x,y;\, u_\lambda;\, u_{\lambda,x},\, u_{\lambda,y}), \\ u_\lambda(x, \varphi(x)) = u_{\lambda,x}(x, \varphi(x)) = u_{\lambda,y}(x, \varphi(x)) = 0. \end{cases}$$

That the sequences $\{u_\lambda\}$, $\{u_{\lambda,x}\}$, $\{u_{\lambda,y}\}$ are uniformly bounded on R, and that the sequence $\{u_\lambda\}$ is equicontinuous on R is immediately evident from the equivalent integral expressions

$$(3.5) \quad u_\lambda(x,y) = \int_{\psi(y)}^{x} d\xi \int_{\varphi(\xi)}^{y} g_\lambda(\xi,\eta;\, u_\lambda;\, u_{\lambda,x}, u_{\lambda,y})\, d\eta$$
$$= \int_{\varphi(x)}^{y} d\eta \int_{\psi(\eta)}^{x} g_\lambda(\xi,\eta;\, u_\lambda;\, u_{\lambda,x}, u_{\lambda,y})\, d\xi,$$

$$(3.6) \quad u_{\lambda,x}(x,y) = \int_{\varphi(x)}^{y} g_\lambda(x,\eta;\, u_\lambda;\, u_{\lambda,x}, u_{\lambda,y})\, d\eta,$$

$$(3.7) \quad u_{\lambda,y}(x,y) = \int_{\psi(y)}^{x} g_\lambda(\xi,y;\, u_\lambda;\, u_{\lambda,x}, u_{\lambda,y})\, d\xi.$$

We now establish the equicontinuity of $\{u_{\lambda x}\}$ and of $\{u_{\lambda y}\}$. This done, the same arguments as those for the proof of Theorem 1a will serve to obtain a subsequence $\{u_\lambda^*\}$ of $\{u_\lambda\}$ which converges uniformly to the solution u.

There is no loss in generality in restricting ourselves at this point to the consideration of those points $(x,y) \in R_2 : \begin{cases} 0 \leq x \leq \ell_1 \\ \varphi(x) \leq y \leq \ell_2 \end{cases}$. For we shall see that the arguments developed below will apply as well for $(x,y) \in R_1 : \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \varphi(x) \end{cases}$ after a simple coordinate translation and rotation. Thus if we insure existence of a solution on R_2, existence on R_1 is simultaneously verified. Moreover, the Cauchy initial data insure that such integral surfaces have a first order contact along γ and hence define an integral surface throughout all of $R = R_1 + R_2$.

Given points $(x_2, y_2) \in R_2$, $(x_1, y_1) \in R_2$, it is always possible to label these points in such a way that $(x_1, y_2) \in R_2$. This being done, we have that

$$(3.8) \quad \left| u_{\lambda,x}(x_1,y_2) - u_{\lambda,x}(x_1,y_1) \right| \leq L \left| y_2 - y_1 \right| ,$$

$$(3.9) \quad \left| u_{\lambda,y}(x_2,y_2) - u_{\lambda,y}(x_1,y_2) \right| \leq L \left| x_2 - x_1 \right| .$$

Assuming, without loss, that $y \geq \varphi(x_2) \geq \varphi(x_1)$, we have that

$$(3.10) \quad u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y) = \int_{\varphi(x_2)}^{y} \left[g_\lambda(x_2,\eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) - g_\lambda(x_1,\eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) \right] d\eta + \int_{\varphi(x_1)}^{\varphi(x_2)} g_\lambda(x_1,\eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\eta$$

We operate on the first integral on the right hand side of (3.10) in the manner demonstrated in equation (2.20). We obtain a formula identical with (2.20) except that here the lower limit of integration is $y = \varphi(x_2)$ instead of $y = 0$. For brevity, we omit the formula.

Since

$$(3.11)\quad \left|\int_{\varphi(x_1)}^{\varphi(x_2)} g_\lambda(x_1,\eta;u_\lambda;u_{\lambda,x},u_{\lambda,y})\,d\eta\right| \leq L\left|\varphi(x_2)-\varphi(x_1)\right|, \quad (\lambda=1,2,\cdots)$$

and since $\varphi(x)$ is uniformly continuous on $[0,\ell_1]$, by the same reasoning as before we arrive at the slight modification to Lemma 2,

$$(3.12)\quad \left|u_{\lambda,x}(x_2,y)-u_{\lambda,x}(x_1,y)\right| \leq K\int_{\varphi(x_2)}^{y}\left|u_{\lambda,x}(x_2,\eta)-u_{\lambda,x}(x_1,\eta)\right|d\eta+\mu+5$$

from which, by Lemma 1,

$$(3.13)\quad \left|u_{\lambda,x}(x_2,y)-u_{\lambda,x}(x_1,y)\right| \leq (\mu+5)e^{k(y-\varphi(x_2))} \leq (\mu+5)e^{k\ell_2}\ .$$

The equicontinuity of $\{u_{\lambda,x}\}$ is thus assured.

The argument for the equicontinuity of $\{u_{\lambda,y}\}$ is similar, hence Theorem 4a obtains.

Remarks c) and d) to Theorem 4 apply as well to Theorem 4a. Quite obviously, if f is continuous, bounded and Lipschitzian (or partially Lipschitzian) on the infinite cylinder with cross section R, then hypothesis 3) of Theorem 4 (or 4a) is immediately satisfied. In fact, this was the form of Theorem 4 which was utilized in the proof of Theorem 4a.

As previously mentioned, the extension of Theorem 4 to systems of equations was first obtained, in principle, by O. NICCOLETTI [14]. He was not, however, explicit about the domain of existence of the solution. The following statement may be derived

from the same arguments of E. KAMKE [2] p. 405 and p. 410 used as the basis for Theorem 4.

Theorem 5.

1) $f_i(x,y; u_1, \cdots, u_n; p_1, \cdots, p_n, q_1, \cdots, q_n) \in C(B'')$

$$B'': \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \\ -a \leq u_i \leq a \\ -b_1 \leq p_i \leq b_1 \\ -b_2 \leq q_i \leq b_2 \end{cases} \qquad (i = 1, \cdots, n).$$

2) The f_i are Lipschitzian on B'', (as defined in Theorem 3).

3) $M \ell_1 \ell_2 \leq a$, $M \ell_1 \leq b_2$, $M \ell_2 \leq b_1$, where

$$M = \max \{ |f_1|, \cdots, |f_n| \} \text{ on } B''.$$

4) $\Upsilon : \begin{cases} 0 \leq x \leq \ell_1 \\ y = \varphi(x) \end{cases}$ where $\varphi(x) \in C'([0,\ell_1])$, $\varphi'(x) \neq 0$

for $x \in [0,\ell_1]$ and $\varphi(0) = \ell_2$, $\varphi(\ell_1) = 0$.

⟹ 5) There exists one and only one set of functions $\{u_1, \cdots, u_n\}$, $u_i(x,y) \in C'(R)$, $u_{i,xy}(x,y) \in C(R)$, $(i = 1, \cdots, n)$, where $R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$, such that for each $(x,y) \in R$ the point $(x,y; u_j(x,y); u_{j,x}(x,y), u_{j,y}(x,y)) \in B$, and

$$u_{i,xy}(x,y) = f_i(x,y; u_j(x,y), u_{j,x}(x,y), u_{j,y}(x,y)),$$

$$u_i(x, \varphi(x)) = u_{i,x}(x, \varphi(x)) = u_{i,y}(x, \varphi(x)) = 0,$$

$(i = 1, \cdots, n)$, for each $(x,y) \in R$.

We may extend the arguments in the proof of Theorem 4a to apply to systems of equations. The extension is practically identical with the previous extension of Theorem 1a to Theorem 3a, except that now Theorem 5 is used to establish existence and uniqueness of the solutions of the system

$$u_{i\lambda,xy} = g_{i\lambda}(x,y;\, u_{j\lambda};\, u_{j\lambda,x},\, u_{j\lambda,y}),\ (i=1,\cdots,n),$$

$$(\lambda = 1,2,\cdots),$$

under the Cauchy initial conditions. We obtain the following theorem:

Theorem 5a

1)

2)' the f_i are partially Lipschitzian on B'', (as defined in Theorem 3a).

3)

4)

$\Longrightarrow$ 5)' There exists at least one set of functions $\{u_1,\cdots,u_n\}$, $u_i(x,y) \in C'(R)$, $u_{i,xy}(x,y) \in C(R)$, $(i = 1,\cdots,n)$, where $R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$, such that for each $(x,y) \in R$ the point $(x,y;\, u_j(x,y);\, u_{j,x}(x,y),\, u_{j,y}(x,y)) \in B$, and
$u_{i,xy}(x,y) = f_i(x,y;\, u_j(x,y);\, u_{j,x}(x,y),\, u_{j,y}(x,y))$,
$u_i(x,\varphi(x)) = u_{i,x}(x,\varphi(x)) = u_{i,y}(x,\varphi(x)) = 0$,
$(i = 1,\cdots, n)$, for each $(x,y) \in R$.

Remark c), with obvious modifications, and Remark d) to Theorem 4 apply as well as to Theorems 5 and 5a. Moreover, in Theorem 5 (or 5a) we may eliminate hypothesis 3) by demanding that the f_i be continuous, bounded and Lipschitzian (or partially Lipschitzian) on the infinite cylinder with cross section R.

CHAPTER IV

Existence Theorems for Canonical Hyperbolic First Order Systems

In this chapter, and in Chapters 5 and 6 as well, we shall not give explicit domains of definition for the functions involved in the differential equations. As a consequence, existence will be shown "in the small" only. We do this because our method of attack will not yield any improvement upon the domains of existence, no matter how large the domains of definition are taken, provided the other hypotheses are not weakened. We shall elaborate on this peculiarity in the course of the exposition.

Theorems 6 and 7 below were given by M. CINQUINI-CIBRARIO [12] p. 180 in the form stated. A statement under somewhat weaker hypotheses, but based on the same proof, may be found in R. COURANT-D. HILBERT [17] p. 324. We examine the proof to show that the arguments therein are independent of the uniqueness of the solutions to the problems involved. As a consequence, our results in Chapters 2 and 3 apply and we arrive at the improved statements given by Theorems 6a and 7a, where hypothesis 2) of Theorems 6 and 7 is dropped altogether and the corresponding conclusions are altered to read "at least one".

Common hypothesis 1) We shall suppose the functions a_{ik}, c_i, $(i,k=1,\cdots,n)$, of arguments $x,y,u_1,\cdots,u_n$, to be continuously differentiable with bounded derivatives in a certain domain D. Fur-

ther, we suppose the determinant

$$|a_{ik}| \neq 0 \quad \text{in } D. \tag{4.1}$$

Under these assumptions, the system

$$\begin{cases} A_i(x,y) = \sum_{k=1}^{n} a_{ik}\, u_{k,x}(x,y) - c_i = 0, & (i=1,\cdots,m<n) \\ B_i(x,y) = \sum_{k=1}^{n} a_{ik}\, u_{k,y}(x,y) - c_i = 0, & (i=m+1,\cdots,n) \end{cases} \tag{4.2}$$

is called a canonical hyperbolic first order system.

Theorem 6. (Characteristic initial value problem.)

1)

2) All first derivatives of the functions a_{ik}, c_i, $(i,k=1,\cdots,n)$ satisfy a Lipschitz condition with respect to arguments $u_1,\cdots,u_n$ in D.

3) $$\left.\begin{array}{l} U_i(x) \in C'([0,\ell_1]) \\ V_i(y) \in C'([0,\ell_2]) \\ U_i(0) = V_i(0) \end{array}\right\} \qquad (i=1,\cdots,n)$$

Moreover, for each $x \in [0,\ell_1]$, the point $(x,0;U_j(x))$[3] $\in D$ and

$$\sum_{k=1}^{n} a_{ik}(x,0;\, U_j(x))U'_k(x) - c_i(x,0;U_j(x)) = 0, \tag{4.3}$$

$$(i=1,\cdots,m<n);$$

and, for each $y \in [0,\ell_2]$, the point $(0,y;\, V_j(y)) \in D$ and

$$\sum_{k=1}^{n} a_{ik}(0,y;V_j(y))V'_k(y) - c_i(0,y;\, V_j(y)) = 0, \tag{4.4}$$

$$(i=m+1,\cdots,n).$$

---

[3] Recall the notation: $(x,y;U_j(x)) = (x,y;U_1(x),\cdots,U_n(x))$.

$\Longrightarrow$ 4) There exists one and only one set of functions $\{u_1,\cdots,u_n\}$, $u_i(x,y) \in C'(R_\eta)$, $u_{i,xy} \in C(R_\eta)$, $(i = 1,\cdots,n)$, where $R_\eta : \begin{cases} 0 \leq x \leq \eta \ell_1 \\ 0 \leq y \leq \eta \ell_2 \end{cases}$, with $0 < \eta \leq 1$ and η sufficiently small, such that the set of functions satisfies the system (4.2) for each $(x,y) \in R_\eta$ and satisfies the conditions

$$\left.\begin{array}{ll} u_i(x,0) = U_i(x) & \text{for } x \in [0, \ \ell_1] \\ u_i(0,y) = V_i(y) & \text{for } y \in [0, \ \ell_2] \end{array}\right\} (i = 1,\cdots,n).$$

Theorem 6a.

1)

3)

$\Longrightarrow$ 4)' There exists at least one set of functions, etc. (as in Theorem 6).

Theorem 7. (Cauchy problem.)

1)

2) (as in Theorem 6.)

5) $\Upsilon : \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases}$ for $\tau \in [0,1]$, $x(\tau)$ and $y(\tau) \in C'([0,1])$ and strictly monotone, i.e., $\dot{x} \neq 0$, $\dot{y} \neq 0$ on $[0,1]$.

$U_i(\tau) \in C'([0,1])$, $(i = 1,\cdots,n)$. For each $\tau \in [0,1]$, the point $(x(\tau), y(\tau); U_j(\tau)) \in D$.

$\Longrightarrow$ 6) There exists one and only one set of functions $\{u_1,\cdots,u_n\}$, $u_i(x,y) \in C'(R_\Upsilon)$, $u_{i,xy}(x,y) \in C(R_\Upsilon)$, $(i = 1,\cdots,n)$, where R_Υ is a sufficiently small neighborhood of the curve Υ, such that

the set of functions satisfies the system (4.2) for each $(x,y) \in R_\gamma$ and satisfies the conditions

$$u_i(x(\tau), y(\tau)) = U_i(\tau) \quad \text{for } \tau \in [0,1], \ (i = 1,\cdots,n).$$

Theorem 7a

1)

5)

$\Rightarrow$ 6)' There exists at least one set of functions etc. (as in Theorem 7.)

The proofs of these theorems are contained in the following argument:

Suppose we have a set of functions $\{u_1,\cdots,u_n\}$, either as a solution to the characteristic initial value problem above on a domain R_η, or as a solution to the Cauchy problem above on a domain R_γ. Then for either case,

$$(4.5) \quad A_{i,y} = \sum_{k=1}^{n} a_{ik} u_{k,xy} + \sum_{k=1}^{n} \left[a_{ik,y} + \sum_{r=1}^{n} \frac{\partial a_{ik}}{\partial u_r} u_{r,y} \right] u_{k,x} - c_{i,y} - \sum_{k=1}^{n} \frac{\partial c_i}{\partial u_k} u_{k,y} = 0, \ (i = 1,\cdots,m<n),$$

$$(4.6) \quad B_{i,x} = \sum_{k=1}^{n} a_{ik} u_{k,xy} + \sum_{k=1}^{n} \left[a_{ik,x} + \sum_{r=1}^{n} \frac{\partial a_{ik}}{\partial u_r} u_{r,x} \right] u_{k,y} - c_{i,x} - \sum_{k=1}^{n} \frac{\partial c_i}{\partial u_k} u_{k,x} = 0, \ (i = m+1,\cdots, n).$$

Equations (4.5) and (4.6) are n linear algebraic equations in the

n unknowns $u_{i,xy}$. Since the determinant of this system, $|a_{ik}|$, does not vanish over the domain in question, we may solve the system to obtain explicitly

$$(4.7) \quad u_{i,xy} = f_i(x,y;\, u_j;\, u_{j,x},\, u_{j,y}),\ (i = 1,\cdots,n).$$

Under hypothesis 1) alone, the f_i are continuous and partially Lipschitzian over any bounded domain in the $3n + 2$ dimensional $(x,y;\, u_j;\, u_{j,x},\, u_{j,y})$-space where $(x,y;\, u_j) \in D$. If hypothesis 2) also applies, the f_i are "fully" Lipschitzian over any such domain.

Consider Theorems 6 and 6a. The characteristic initial conditions imposed therein, coupled with the system (4.7), form a problem of the type considered in Theorems 3 and 3a, respectively. (Chapter 2). We have shown above that any solution of a canonical hyperbolic system is also a solution of a particular system of type (4.7). If we now demonstrate the converse for characteristic initial conditions, i.e. that any solution of the derived system (4.7) is also a solution of the original system (4.2), then Theorems 6 and 6a follow directly from Theorems 3 and 3a respectively.

Suppose we have a set of functions $\left\{u_1,\cdots,u_n\right\}$ as a solution of (4.7) over a certain domain including the initial characteristics. By (4.5) and (4.6), which are merely alternative forms of (4.7), we havd

$$(4.8)\quad \begin{cases} A_{i,y}(x,y) = 0 \quad , \quad (i = 1,\cdots, m < n) \\ B_{i,x}(x,y) = 0 \quad , \quad (i = m+1,\cdots, n) \end{cases}$$

over this domain. But, by (4.3) and (4.4) of hypothesis 3) to both Theorems 6 and 6a, we have that

$$(4.9)\quad \begin{cases} A_i(x,0) = 0 \quad , \quad (i = 1,\cdots, m < n) \\ B_i(0,y) = 0 \quad , \quad (i = m+1,\cdots n), \end{cases}$$

whence

$$A_i(x,y) \equiv 0 \quad , \quad (i = 1,\cdots, m < n),$$
$$B_i(x,y) \equiv 0 \quad , \quad (i = m+1,\cdots, n),$$

throughout the domain. Hence the converse is shown.

For the Cauchy problem considered in Theorems 7 and 7a, we observe first that we can determine $u_{i,x}(x(\tau), y(\tau))$ and $u_{i,y}(x(\tau), y(\tau))$, $(i = 1,\cdots,n)$, as functions continuous for each $\tau \in [0,1]$, solely from the prescription of $u_i(x(\tau), y(\tau)) = U_i(\tau)$, $(i = 1,\cdots,n)$, and the requirement that the canonical hyperbolic system (4.2) must be satisfied at each point of Υ. For, since $\dot{x} + \dot{y}^2 \not\leq 0$ along Υ, we may write the strip conditions

$$(4.10)\qquad \dot{u}_i = p_i\dot{x} + q_i\dot{y}, \quad (i = 1,\cdots,n),$$

as one of

$$(4.11)\quad q_i = \frac{1}{\dot{y}}(\dot{u}_i - p_i\dot{x}) \quad \text{or } p_i = \frac{1}{\dot{x}}(\dot{u}_i - q_i\dot{y}), \ (i = 1,\cdots,n).$$

Consider a particular point $P \in \Upsilon$ where $\dot{y} \not\leq 0$. Here we substitute $q_i = u_{i,y} = \frac{1}{\dot{y}}(\dot{u}_i - p_i\dot{x})$ into equations $B_i(P) = 0$, $(i=m+1,\cdots,n)$. These, together with the equations $A_i(P) = 0$, $(i = 1,\cdots, m<n)$,

form a linear algebraic system in the $p_i = u_{i,x}(P)$ with determinant $|a_{ik}| \neq 0$. Thus the p_i are uniquely determined at P, and, by (4.11), the q_i as well are uniquely determined at P. If $\dot{y} = 0$ at P, then $\dot{x} \neq 0$ there and a similar argument applies utilizing $p_i = \frac{1}{\dot{x}}(\dot{u}_i - q_i\dot{y})$.

Thus we have, in effect, prescribed all three sets u_i, $u_{i,x}$, $u_{i,y}$, $(i = 1,\cdots,n)$, along γ once the u_i are prescribed along γ and the $u_{i,x}$ and the $u_{i,y}$ are merely required to satisfy the strip conditions (4.10) and the canonical hyperbolic system at (4.2) at each point of γ.

Suppose we have a set of functions $\left\{u_1,\cdots, u_n\right\}$ as a solution of

(4.7) $u_{i,xy} = f_i(x,y; u_j; u_{j,x}, u_{j,y})$, $(i = 1,\cdots,n)$ in a neighborhood of the initial curve γ and taking, with their first derivatives, precisely the above determined values at each point of γ. Then by (4.5) and (4.6), the fact that these functions and their first derivatives satisfy the canonical hyperbolic system (4.2) at each point of γ implies further that the system (4.2) corresponding to (4.7) is satisfied everywhere in the neighborhood in question.

With hypothesis 2) imposed, system (4.7) and the initial data on γ satisfy the hypotheses of Theorem 5, while without hypothesis 2), system (4.7) and the initial data on γ satisfy the hypotheses of Theorem 5a. But since we have shown above that each solution of (4.7) is a solution of the corresponding canonical

hyperbolic system (4.2), we have that Theorem 7 is a consequence of Theorem 5, while Theorem 7a is a consequence of Theorem 5a.

In these four theorems we are unable to prescribe the domain of definition of the functions

$$f_i(x,y;\, u_j;\, p_j,q_j),\ (i = 1,\cdots, n),$$

in such a way as to insure existence of a solution throughout $R: \begin{cases} 0 \leq x \leq \ell_1 \\ 0 \leq y \leq \ell_2 \end{cases}$. This is because the f_i are continuous for all p_j and q_j, $(j = 1,\cdots,n)$, but may turn out to be bounded only when these variables are restricted to finite domains. The following example demonstrates why the existence of solutions can be found only "in the small".

Example 3. Consider the characteristic initial value problem for the system

$$\begin{array}{lll} u_{1,xy} = u_{1,x}^2 \;, & u_1(x,-1) = x, & u_1(0,y) = 0 \\ u_{2,xy} = 0 \;, & u_2(x,-1) = 0, & u_2(0,y) = 0 \\ \vdots & \vdots & \\ u_{n,xy} = 0 \;, & u_n(x,-1) = 0, & u_n(0,y) = 0. \end{array}$$

By quadratures, we obtain the solution $u_1(x,y) = \frac{-x}{y}$, while $u_2 = \cdots = u_n = 0$, quite obviously. The f_i corresponding to this problem possess derivatives of all orders for all values of all variables. However, $f_1 = u_{1,x}^2$ becomes unbounded as the argument $u_{1,x}$ increases indefinitely in absolute value. We note that, despite the specification of initial data everywhere along the

intersecting characteristics $x = 0$ and $y = -1$, the first function in the solution, namely u_1, has a discontinuity across the line $y = 0$. Hence this example typifies those cases for which solutions exist "in the small" only.

We note that Remark d) of Chapter III applies as well to hypothesis 5) of Theorems 7 and 7a. The statement is that $\Upsilon : \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases}$ for $\tau \in [0,1]$ need only have $x(\tau)$ and $y(\tau) \in C'([0,1])$, monotone, and with $\dot{x}^2 + \dot{y}^2 \neq 0$ at each point of Υ. In fact, the argument in the proof above applies directly to this statement.

CHAPTER V.

The Cauchy Problem for $F(x,y;\ u;\ p,q;\ r,s,t) = 0$.

In this chapter we concern ourselves with the Cauchy problem for the general non-linear second order partial differential equation in the hyperbolic domain. Specifically, the problem is to determine an integral surface of the equation

$$(1.1)\qquad F(x,y;\ u;\ p,q;\ r,s,t) = 0$$

such that the hyperbolic condition

$$(1.3)\qquad F_s^{\,2} - 4\,F_r\,F_t > 0$$

is satisfied thereon; moreover, the integral surface must have a second order contact with a given second order strip which is nowhere a characteristic strip.

In his celebrated paper [10], H. LEWY successfully attacks this problem by reducing equation (1.1) to a system of first order partial differential equations for the unknowns $x,y;\ u;\ p,q;\ r,s,t$ as functions of the parameters λ and μ of the two families of characteristics on the integral surface in question. LEWY's existence proof for the system is based on a finite difference argument. However, the system is of canonical hyperbolic form and the theorem of M. CINQUINI-CIBRARIO, Theorem 7 of Chapter IV, is immediately applicable and insures existence and uniqueness of the solution in a sufficiently small neighborhood of the initial strip. Moreover, as demonstrated below, Theorem 7a may be used to effect an improvement on LEWY's work.

We present simultaneously LEWY's original theorem and our

improvement on it. LEVY's theorem is obtained by omitting the parentheses. Our theorem is obtained by replacing the underscored statements by the corresponding ones in the parentheses.

Theorem 8 (8a)

1) $$S^2: \begin{cases} x = x(\tau) \\ y = y(\tau) \\ u = u(\tau) \\ p = p(\tau) \\ q = q(\tau) \\ r = r(\tau) \\ s = s(\tau) \\ t = t(\tau) \end{cases}$$ for $\tau \in [0,1]$ is a nowhere characteristic second order strip,

i.e. $x, y;\ u;\ p, q;\ r, s, t(\tau) \in C'([0,1])$, and for each $\tau \in [0,1]$,

i) $\dot{x}^2 + \dot{y}^2 \neq 0$,

ii) $F_r\, \dot{y}^2 - F_s\, \dot{y}\, \dot{x} + F_t\, \dot{x}^2 \neq 0$,

iii) $F_s^{\,2} - 4\, F_r\, F_t > 0$,

iv) $F(x(\tau), y(\tau);\ u(\tau);\ p(\tau), q(\tau);\ r(\tau), s(\tau), t(\tau))$
$= 0$.

2) $F \underline{\in C'''}$ ($\underline{\in C''}$) in a certain neighborhood of S^2.

3) There exists one and only one (at least one) integral surface $J: u = u(x,y)$ of the equation $F(x,y;\ u;\ p,q;\ r,s,t) = 0$ such that $u(x,y) \in C'''$ in a sufficiently small neighborhood of the base curve $\Upsilon : \begin{cases} x = x(\tau) \\ y = y(\tau) \end{cases}$ for $\tau \in [0,1]$, and such that $J: u = u(x,y)$ has a second order contact with the strip S^2.

Proof

We first demonstrate that any solution of the above problem, together with its derivatives of the first and second orders, represents a solution of a particular canonical hyperbolic system under the same boundary conditions.

We assume that $F_r \neq 0$ and $F_t \neq 0$ in the domains considered in the following argument. This may be done without loss of generality. For, by Definition 1a, a characteristic base curve must satisfy

$$(1.5) \quad \begin{array}{ll} 1) & F_r \dot{y}^2 - F_s \dot{y} \dot{x} + F_t \dot{x}^2 = 0, \\ 2) & \dot{x}^2 + \dot{y}^2 \neq 0 . \end{array}$$

Suppose at a point of S^2 that $F_r = 0$. Then $\dot{x} = 0$ represents the vertical tangent taken by one of the characteristic base curves through the projection of this point onto the xy plane. Conversely, if one of the characteristic base curves through a point in the projection of S^2 has a vertical tangent, then $\dot{x} = 0$ there and, consequently, $F_r = 0$ at the corresponding point on S^2. Likewise, $F_t = 0$ if and only if $\dot{y} = 0$, in the sense above. Thus, by a suitable coordinate rotation in the xy plane, we may insure that $F_r \neq 0$ and $F_t \neq 0$ in a neighborhood of the point in question on S^2. Granting that this is a local property only and that the particular rotation performed may introduce values of $F_r = 0$ or $F_t = 0$ at some other sufficiently distant points on S^2, we observe that this local property is sufficient because our proof is ultimately based upon Theorems 4 and 4a of Chapter III. In those

theorems the integral equation statement of the problem made it plainly evident that the value of the integral at any point P depended only upon the portion of the initial curve cut off by the two characteristics intersecting at P. Consequently, we may consider the arguments below as applying in succession to small overlapping segments of S^2, with coordinate axes rotated suitably for each segment considered. (See also R. COURANT - D. HILBERT [17] p. 323 and p. 332.)

Let us assume that we have an integral surface J: $u=u(x,y)$ satisfying the conditions of either Theorem 8 or Theorem 8a. Then by (1.5) we conclude that the related characteristic base curves are the two one-parameter families of curves determined by the equations

$$y_\lambda = \rho_1 x_\lambda \,, \tag{5.1}$$

$$y_\mu = \rho_2 x_\mu \,, \tag{5.2}$$

where

$$\rho_1 = \frac{F_s + \sqrt{F_s^2 - 4 F_r F_t}}{2 F_r} \,, \tag{5.3}$$

$$\rho_2 = \frac{F_s - \sqrt{F_s^2 - 4 F_r F_t}}{2 F_r} \,. \tag{5.4}$$

ρ_1 and ρ_2 are functions of the variables x,y; u; p,q; r,s,t and $\rho_1 \neq \rho_2$ in a neighborhood of S^2 by the hyperbolic condition (1.3).

Consider the coordinate transformation

$$\left\{ \begin{array}{l} x = x(\lambda, \mu) \\ y = y(\lambda, \mu) \,. \end{array} \right. \tag{5.5}$$

The Jacobian of this transformation,

$$(5.6)\qquad y_\lambda x_\mu - y_\mu x_\lambda = (\rho_1 - \rho_2) x_\lambda x_\mu ,$$

does not vanish in a vicinity of the projection of S^2. This follows since $\rho_1 \neq \rho_2$; while $x_\lambda = 0$ would, by (5.1), imply $y_\lambda = 0$, contradicting the requirement $\dot{x}^2 + \dot{y}^2 \neq 0$, (similarly for x_μ). Hence the inverse transformation,

$$(5.7)\qquad \begin{cases} \lambda = \lambda(x,y) \\ \mu = \mu(x,y) \end{cases} ,$$

exists in a vicinity of the projection of S^2.

Along the characteristics on J: u=u(x,y) certain additional equations must be satisfied. These are determined as follows:

Since $F \in C'''(\in C'')$ and $u \in C'''$, we obtain by differentiation

$$(5.8)\qquad \begin{cases} F_r r_x + F_s s_x + F_t t_x = -[F]_x \\ x_\lambda r_x + y_\lambda s_x = r_\lambda \\ x_\lambda s_x + y_\lambda t_x = s_\lambda , \end{cases}$$

where

$$(5.9)\qquad [F]_x = F_p r + F_q s + F_u p + F_x .$$

Similarly,

$$(5.10)\qquad \begin{cases} F_r r_y + F_s s_y + F_t t_y = -[F]_y \\ x_\lambda r_y + y_\lambda s_y = s_\lambda \\ x_\lambda s_y + y_\lambda t_y = t_\lambda , \end{cases}$$

where

(5.11) $$[F]_y = F_p s + F_q t + F_u q + F_y \, .$$

Since λ is the parameter for one family of characteristic curves and, consequently, is the path parameter along each of the curves of the other family, the determinant

(5.12) $$\begin{vmatrix} F_r & F_s & F_t \\ x_\lambda & y_\lambda & 0 \\ 0 & x_\lambda & y_\lambda \end{vmatrix} = F_r y_\lambda^2 - F_s y_\lambda x_\lambda + F_t x_\lambda^2 = 0.$$

Hence the quantities on the right-hand side in each of the systems (5.8) and (5.10) must be linearly dependent, i.e. in each system the augmented matrix of coefficients must be of rank less than three. Consequently,

(5.13) $$\begin{vmatrix} F_r & F_t & [F]_x \\ x_\lambda & 0 & -r_\lambda \\ 0 & y_\lambda & -s_\lambda \end{vmatrix} = F_r r_\lambda y_\lambda + F_t s_\lambda x_\lambda + [F]_x x_\lambda y_\lambda = 0.$$

Recalling the assumption made without loss,

$x_\lambda = \frac{1}{\rho_1} y_\lambda$ and $y_\lambda \not\equiv 0$, equation (5.13) reduces to

(5.14) $$F_r r_\lambda + \frac{1}{\rho_1} F_t s_\lambda + [F]_x \, x_\lambda = 0.$$

Likewise, from (5.10) we obtain the linear dependence of the right-hand terms in the form

(5.15) $$\rho_1 F_r s_\lambda + F_t t_\lambda + [F]_y \, y_\lambda = 0.$$

Along the curves of the other family of characteristics the following relations must be satisfied. These are obtained in a

fashion completely analogous to that used in obtaining (5.14) and (5.15):

$$F_r r_\mu + \frac{1}{\rho_2} F_t s_\mu + [F]_x x_\mu = 0 \tag{5.16}$$

$$\rho_2 F_r s_\mu + F_t t_\mu + [F]_y y_\mu = 0. \tag{5.17}$$

In addition, the strip conditions

$$\dot u = p \dot x + q \dot y \tag{1.8}$$

$$\left\{ \begin{aligned} \dot p &= r \dot x + s \dot y \\ \dot q &= s \dot x + t \dot y \end{aligned} \right. \tag{1.9}$$

must be satisfied along any curve lying on J: u=u(x,y). In particular, they must be satisfied along any characteristic on J.

From equations (5.1), (5.2), (5.14) through (5.17), (1.8) and (1.9) we obtain the following system of "characteristic equations" i.e. equations which must be satisfied along the characteristics on any integral surface J:

$$\left. \begin{aligned} \varphi_1 &= y_\lambda - \rho_1 x_\lambda = 0 \\ \varphi_2 &= F_r r_\lambda + \frac{1}{\rho_1} F_t s_\lambda + [F]_x x_\lambda = 0 \\ \varphi_3 &= \rho_1 F_r s_\lambda + F_t t_\lambda + [F]_y y_\lambda = 0 \\ \varphi_4 &= u_\lambda - p x_\lambda - q y_\lambda = 0 \\ \varphi_5 &= p_\lambda - r x_\lambda - s y_\lambda = 0 \\ \varphi_6 &= q_\lambda - s x_\lambda - t y_\lambda = 0 \\ \\ \psi_1 &= y_\mu - \rho_2 x_\mu = 0 \\ \psi_2 &= F_r r_\mu + \frac{1}{\rho_2} F_t s_\mu + [F]_x x_\mu = 0 \end{aligned} \right\} \begin{array}{c} \text{System} \\ \text{A} \end{array} \tag{5.18}$$

(5.18)
(continued)

$$\left.\begin{aligned} \psi_3 &= \rho_2 F_r s_\mu + F_t t_\mu + [F]_y\, y_\mu = 0 \\ \psi_4 &= u_\mu - p\, x_\mu - q\, y_\mu = 0 \\ \psi_5 &= p_\mu - r\, x_\mu - s\, y_\mu = 0 \\ \psi_6 &= q_\mu - s\, x_\mu - t\, y_\mu = 0 \end{aligned}\right\}\text{System B}$$

We observe that System A of (5.18) is of canonical hyperbolic form in x,y; u; p,q; r,s,t as functions of λ and μ. Since for Theorem 8, $F \in C'''$, while for Theorem 8a, $F \in C''$, the coefficients of all equations in (5.18) are functions of class C'' for Theorem 8, and of class C' for Theorem 8a. Moreover, the determinant of the matrix of coefficients for System A, is, after interchange of rows and columns,

$$(5.19) \qquad \begin{vmatrix} -\rho_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\rho_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & F_r & \frac{1}{\rho_1}F_t & 0 & 0 & 0 & 0 \\ 0 & * & 0 & \rho_1 F_r & F_t & 0 & 0 & 0 \\ * & 0 & F_r & \frac{1}{\rho_2}F_t & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 1 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 1 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= F_r F_t^2 \cdot \left(\frac{\rho_1 - \rho_2}{\rho_1 \rho_2}\right)^2,$$

where the coefficients designated only by asterisks, *, do not contribute to the value of the determinant. Since $F_r \neq 0$, $F_t \neq 0$ and $\rho_1 \neq \rho_2$ in a neighborhood of S^2, the determinant (5.19) does not vanish therein. Hence any solution J: u=u(x,y) of the problem of Theorem 8, together with its first and second derivatives,

satisfies the hypotheses for Theorem 7; because the requirement that $F \in C'''$ is certainly sufficient to insure that the first derivatives of the coefficients of System A be Lipschitzian with respect to variables x,y; u; p,q; r,s,t. Moreover, the requirement in Theorem 8a that $F \in C''$ insures that the coefficients of System A are of class C', as demanded by Theorem 7a.

In the $\lambda\mu$, or characteristic, plane, the initial base curve has the parametric form

$$\Upsilon : \begin{cases} \lambda = \lambda(x(\tau), y(\tau)) & \text{for } \tau \in [0,1], \\ \mu = \mu(x(\tau), y(\tau)) \end{cases}$$

and is nowhere parallel to either the λ or μ axes. Consequently, Υ may be expressed in the non-parametric form

$$\lambda = \varphi(\mu)$$

where $\varphi(\mu) \in C'$ and $\varphi'(\mu) \neq 0$. If we introduce $\lambda' = \lambda$ and $\mu' = -\varphi(\mu)$ as new characteristic parameters, we observe that equations (5.18) remain unaltered in form. Hence we may assume, without loss, that the initial base curve Υ has the representation

$$\lambda + \mu = 0 \tag{5.20}$$

in the $\lambda\mu$ plane.

We now demonstrate that any solution of System A satisfying the given Cauchy initial conditions is also a solution of the problem of Theorems 8 and 8a. This done, Theorems 8 and 8a are immediate consequences of Theorems 7 and 7a, respectively.

Following J. HADAMARD [11] p. 504, we show that for each set of functions satisfying System A and the initial conditions on

$\lambda + \mu = 0$, the System B is likewise satisfied. Note that in this part of the argument we cannot admit that p,q,r,s and t are derivatives of u. This is now a matter of proof.

Differentiating $F(x,y;\ u;\ p,q;\ r,s,t)$ by λ and observing equations (5.18), we obtain

$$(5.21) \qquad \frac{dF}{d\lambda} = \varphi_2 + \varphi_3 + F_u\,\varphi_4 + F_p\,\varphi_5 + F_q\,\varphi_6.$$

Hence $\frac{dF}{d\lambda} = 0$ for each set of functions satisfying System A. However, by hypothesis, $F = 0$ along $\lambda + \mu = 0$. Thus $F \equiv 0$ throughout that region where the set of functions satisfying System A is defined. This in turn implies that

$$(5.22) \qquad \frac{dF}{d\mu} = \psi_2 + \psi_3 + F_u\,\psi_4 + F_p\,\psi_5 + F_q\,\psi_6 = 0$$ throughout the same region. By hypothesis, $\psi_2 = 0$ in this region, hence

$$(5.23) \qquad \psi_3 = -F_u\,\psi_4 - F_p\,\psi_5 - F_q\,\psi_6$$

therein.

Since $\rho_1\rho_2 = \frac{F_t}{F_r}$, we obtain from (5.18) by simple algebraic operations

$$(5.24) \qquad \frac{\rho_1\,y_\mu}{F_t}\,\varphi_2 = r_\lambda\,x_\mu + s_\lambda\,y_\mu + H,$$

$$(5.25) \qquad \frac{\rho_2\,y_\lambda}{F_t}\,\psi_2 = r_\mu\,x_\lambda + s_\mu\,y_\lambda + H,$$

where

$$(5.26) \qquad H = \frac{y_\lambda\,y_\mu}{F_t}\,[F]_x = \frac{x_\lambda\,x_\mu}{F_r}\,[F]_x\ ;$$

$$(5.27) \qquad \frac{y_\mu}{F_t}\,\varphi_3 = s_\lambda\,x_\mu + t_\lambda\,y_\mu + K,$$

$$(5.28)\qquad \frac{y_\lambda}{F_t}\,\psi_3 = s_\mu x_\lambda + t_\mu y_\lambda + K,$$

where

$$(5.29)\qquad K = \frac{y_\lambda\, y_\mu}{F_t}\,[F]_y = \frac{x_\lambda\, x_\mu}{F_r}\,[F]_y\,.$$

By Theorem 7 or Theorem 7a, the functions of the set satisfying System A and the Cauchy initial data are continuously differentiable and possess continuous mixed second derivatives. Thus we may perform the differentiations in the following relations:

$$(5.30)\qquad \begin{aligned}\psi_{4,\lambda} - \varphi_{4,\mu} &= p_\lambda x_\mu + q_\lambda y_\mu - p_\mu x_\lambda - q_\mu y_\lambda\\ &= \varphi_5 x_\mu - \varphi_6 y_\mu - \psi_5 x_\lambda - \psi_6 y_\lambda\ ;\end{aligned}$$

$$(5.31)\qquad \begin{aligned}\psi_{5,\lambda} - \varphi_{5,\mu} &= r_\lambda x_\mu + s_\lambda y_\mu - r_\mu x_\lambda - s_\mu y_\lambda\\ &= \frac{\rho_1 y_\mu}{F_t}\,\varphi_2 - \frac{\rho_2 y_\lambda}{F_t}\,\psi_2,\end{aligned}$$

by (5.24) and (5.25) above;

$$(5.32)\qquad \begin{aligned}\psi_{6,\lambda} - \varphi_{6,\mu} &= s_\mu x_\lambda + t_\mu y_\lambda - s_\lambda x_\mu - t_\lambda y_\mu\\ &= \frac{y_\lambda}{F_t}\,\psi_3 - \frac{y_\mu}{F_t}\,\varphi_3\,.\end{aligned}$$

by (5.27) and (5.28) above. But System A is satisfied, hence (5.30), (5.31) and (5.32), by virtue of (5.23), reduce to

$$(5.33)\qquad \left\{\begin{aligned}\psi_{4,\lambda} &= -\,\psi_5 x_\lambda - \psi_6 y_\lambda\\ \psi_{5,\lambda} &= 0\\ \psi_{6,\lambda} &= \frac{-y_\lambda}{F_t}\,(F_u\,\psi_4 + F_p\,\psi_5 + F_q\,\psi_6).\end{aligned}\right.$$

In (5.33) all functions are known except ψ_4, ψ_5, ψ_6 and their derivatives with respect to λ. Moreover, along $\lambda = -\mu$ System B is satisfied, i.e. $\psi_4 = \psi_5 = \psi_6 = 0$ for $\lambda = -\mu$. For fixed μ we may consider (5.33) as a homogeneous system of linear first order ordinary differential equations under homogeneous onepoint boundary conditions. This system has the unique solution

$$\psi_4 = \psi_5 = \psi_6 = 0$$

throughout the region of definition of the set of functions satisfying System A. By (5.23), $\psi_3 = 0$ also, and the System B is shown to be dependent upon the System A in the sense above.

From the functions $x = x(\lambda, \mu)$, $y = y(\lambda, \mu)$ of the set satisfying System A, we may form the inverse functions $\lambda = \lambda(x,y)$, $\mu = \mu(x,y)$, since the Jacobian

(5.6) $$y_\lambda x_\mu - y_\mu x_\lambda = (\rho_1 - \rho_2) x_\lambda x_\mu$$

does not vanish. Hence we may express the function $u = u(\lambda, \mu)$ as a function of the independent variables x and y.

We now need to show only that

(5.34) $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$ and $t = u_{yy}$

throughout the above region to complete the proof.

Now $$\varphi_4 = u_\lambda - px_\lambda - qy_\lambda = 0$$
$$\psi_4 = u_\mu - px_\mu - qy_\mu = 0,$$

while the determinant of this linear system is the Jacobian (5.6) and hence does not vanish. Thus there exists a unique solution.

But $p - u_x$, $q = u_y$ obviously satisfies and hence represents the unique solution.

Similarly,

$$\varphi_5 = u_{x,\lambda} - rx_\lambda - sy_\lambda = 0$$
$$\psi_5 = u_{x,\mu} - rx_\mu - sy_\mu = 0,$$

hence $r = u_{xx}$ and $s = u_{xy}$;

$$\varphi_6 = u_{y,\lambda} - sx_\lambda - ty_\lambda = 0$$
$$\psi_6 = u_{y,\mu} - sx_\mu - ty_\mu = 0,$$

hence $t = u_{yy}$ and $u_{yx} = u_{xy} = s$. The proof is now complete.

CHAPTER VI

The Characteristic Initial Value Problem for $F(x,y;u;p,q;r,s,t) = 0$.

The whole idea of a characteristic initial value problem for the equation

$$F(x,y;\ u;\ p,q;\ r,s,t) = 0 \tag{1.1}$$

appears paradoxical at first glance. In the Cauchy problem the prescribed initial data was sufficient to determine whether or not the projection of the initial curve was characteristic. In this problem, however, we merely prescribe two intersecting space curves through which an integral surface of the equation (1.1) must pass. Since the characteristics are, in general, dependent on the integral surface in question, it would appear impossible to determine, a priori, whether or not the prescribed initial curves have characteristic projections.

That such is not the case is demonstrated by M. CINQUINI-CIBRARIO [13]. In this paper she treats the characteristic initial value problem as a special case of the more general Goursat problem, i.e. where two arbitrary intersecting space curves are prescribed through which an integral surface of (1.1) must pass. Commencing on p. 220, she gives the necessary and sufficient conditions that these curves be characteristic to any integral surface passing through them. We call curves satisfying these conditions "intrinsically characteristic" curves.

In this chapter we examine her development, for the particular case of the characteristic initial value problem, up to the point where a modified form of the system of characteristic equations (5.18) and the above necessary and sufficient conditions are obtained. There are two important differences between her development and that of H. LEWY given in the preceding chapter. First, she transforms the initial curves into the coordinate axes. Since these curves are characteristic, this implies immediately that $F_r = 0$ and $F_t = 0$ at the origin. Thus many of the divisions performed in Chapter V are now invalidated. Second, she is able to solve (1.1) explicitly for s, obtaining

$$s = f(x,y;\ u;\ p,q;\ r,t)$$

and thus to reduce the number of equations in the system of characteristic equations by two.

We do not follow the remainder of her existence proof, in which she reduces the system of characteristic equations to an integral equation form and then applies successive approximations to obtain the existence of a unique solution to the general Goursat problem. Instead we deal directly with the special case of the characteristic initial value problem by a method analogous to that of Chapter V. Such an approach is indicated by M. CINQUINI-CIBRARIO, herself, [12] p.180, footnote 8. She states, in effect, that the following Theorem 9 can be shown to be a consequence of Theorem 6, Chapter IV. We present this proof in detail and, in addition, we extend it to apply to the derivation of Theorem 9a as a consequence of Theorem 6a. The improvement obtained corresponds to that of

Chapter V for the Cauchy problem. Namely, the requirement that $F \in C'''$ is reduced to require merely that $F \in C''$ while the conclusion is altered to read "at least one solution" instead of "one and only one solution".

Theorem 9

1)
$$\Gamma_1 : \begin{cases} \gamma_1 : \begin{cases} x_1 - \xi \leq x \leq x_1 + \xi \\ y = f_1(x) \end{cases} \\ u = F_1(x) \end{cases} \quad , \quad \begin{matrix} f_1(x) \in C''([x_1 - \xi , x_1 + \xi]) \\ F_1(x) \in C''([x_1 - \xi , x_1 + \xi]). \end{matrix}$$

$$\Gamma_2 : \begin{cases} \gamma_2 : \begin{cases} x = f_2(y) \\ y_1 - \eta \leq y \leq y_1 + \eta \end{cases} \\ u = F_2(y) \end{cases} \quad , \quad \begin{matrix} f_2(y) \in C''([y_1 - \eta , y_1 + \eta]) \\ F_2(y) \in C''([y_1 - \eta , y_1 + \eta]) \end{matrix}$$

The point (x_1, y_1) is the only point of intersection of γ_1 and γ_2 and it is interior to both curves. Moreover, $F_1(x_1) = F_2(y_1)$ and $f_1'(x_1) f_2'(y_1) \neq 1$. (i.e. γ_1 and γ_2 do not have a common tangent at the point (x_1, y_1).)

2) Γ_1 and Γ_2 are "intrinsically characteristic" in a neighborhood of their point of intersection, i.e. they meet the necessary and sufficient conditions, given below, that they be characteristic to any integral surface of

(1.1) $\qquad F(x,y;\ u;\ p,q;\ r,s,t) = 0$

passing through them. As we shall see below, this hypothesis, together with hypothesis 1), tacitly implies that at the intersection point (x_1, y_1, u_1) of Γ_1 and Γ_2 the values p_1, q_1, r_1, s_1,

t_1), the hyperbolic condition

$$F_{s_1}^{\,2} - 4\,F_{r_1}\,F_{t_1} > 0,$$

is satisfied, (notation: $F_{s_1} = F_s(x_1,y_1;\ u_1;\ p_1,q_1;r_1,s_1,t_1)$,etc.)

3) $F \in C'''$ in a neighborhood of the point

$$(x_1,y_1;\ u_1;\ p_1,q_1;\ r_1,s_1,t_1).$$

$\Longrightarrow$ 4) There exists one and only one integral surface $J: u=u(x,y)$ of $F(x,y;\ u;p,q;\ r,s,t) = 0$, defined and of class C''' in a sufficiently small neighborhood of the point (x_1,y_1) and passing through subarcs of Γ_1 and Γ_2 intersecting at the point (x_1,y_1,u_1).

Theorem 9a

1)

2)

3)' $F \in C''$ in a neighborhood of the point

$$(x_1,y_1;\ u_1;\ p_1,q_1;r_1,s_1,t_1).$$

$\Longrightarrow$ 4)' There exists at least one integral surface etc. (as in Theorem 9).

Proof of Theorems 9 and 9a

We first perform the coordinate transformation

$$(6.1)\qquad \begin{cases} \bar{x} = x - f_2(y) \\ \bar{y} = y - f_1(x) \end{cases}$$

taking Υ_1 into the $\bar{x}$ axis, Υ_2 into the $\bar{y}$ axis and the point (x_1,y_1) into the origin. This transformation is univalent in a

neighborhood of (x_1,y_1) since the Jacobian

$$(6.2) \qquad 1 - f_1'(x_1)f_2'(y_1) \neq 0$$

by hypothesis 1). Geometrically, this means that γ_1 and γ_2 do not have a common tangent at their point of intersection.

Without loss, we may assume homogeneous initial conditions. For, suppose we have an integral surface J: $u=u(x,y)$ of equation (1.1) passing through the curves Γ_1 and Γ_2. Then by the above transformation, considering (6.2),

$$(6.3) \qquad u(x,y) = \bar{u}(\bar{x}(x,y), \bar{y}(x,y)),$$

and hence for any such integral surface

$$(6.4) \quad \begin{cases} F_1(x) = u(x,f_1(x)) = \bar{u}(\bar{x}(x,f_1(x)), 0), \\ F_2(y) = u(f_2(y),y) = \bar{u}(0,\bar{y}(f_2(y),y)). \end{cases}$$

Letting

$$(6.5) \quad w(\bar{x},\bar{y}) = \bar{u}(\bar{x},\bar{y}) - \bar{u}(\bar{x},0) - \bar{u}(0,\bar{y}) + \bar{u}(0,0),$$

and since, by hypothesis 1), f_1, f_2, F_1 and $F_2 \in C''$, we obtain

$$(6.6) \quad \begin{aligned} w(\bar{x},0) &= w_{\bar{x}}(\bar{x},0) = w_{\bar{x}\bar{x}}(\bar{x},0) = 0, \\ w(0,\bar{y}) &= w_{\bar{y}}(0,\bar{y}) = w_{\bar{y}\bar{y}}(0,\bar{y}) = 0. \end{aligned}$$

Thus we may reduce the problem to that of finding a function $w = w(\bar{x},\bar{y})$ which vanishes on the coordinate axes in a vicinity of the origin and satisfies there the transformed form of equation (1.1),

$$(6.7)\quad F(\overline{x},\overline{y};[w+g];\ [w+g]_{,x},\ [w+g]_{,\overline{y}};\ [w+g]_{,\overline{xx}},\ [w+g]_{,\overline{xy}},\ [w+g]_{,\overline{yy}})$$

where

$$(6.8)\quad g(\overline{x},\overline{y}) = \overline{u}(\overline{x},0) + \overline{u}(0,\overline{y}) - \overline{u}(0,0).$$

The function g is known from the prescribed initial data.

For simplicity, we return to our original notation and state the problem in this way:

To determine the function $u = u(x,y)$ satisfying equation (1.1) and the initial conditions

$$u(x,0) = u(0,y) = 0,$$

where, in the notation above,

$$u_0 = p_0 = q_0 = r_0 = t_0 = 0$$

and

$$(6.9)\quad F(0,0;\ 0;\ 0,0;\ 0,s_0 0) = 0.$$

By hypothesis 2), there exists a unique value s_0 satisfying (6.9).

The characteristic base curves and, a fortiori, the hyperbolic condition are invariant under the transformation (6.1). (See R. COURANT - D. HILBERT [17] p. 304.) Moreover, the substitution $w = \overline{u} - g$ also preserves the invariance of the equation for the characteristic base curves and the hyperbolic condition as is easily seen by differentiation of (6.7). Hence, by hypothesis 2), we have the hyperbolic condition

(6.10) $F_{s_o}^2 - 4 F_{r_o} F_{t_o} > 0,$

while the equation for the characteristic base curve directions at the origin is

(6.11) $$F_{r_o} dy^2 - F_{s_o} dxdy + F_{t_o} dx^2 = 0.$$

Hypothesis 2) implies that the coordinate axes must be characteristic base curves. By (6.11) and (6.10) this in turn implies that $F_{r_o} = F_{t_o} = 0$, and hence that $F_{s_o} \neq 0$. But now the Implicit Function Theorem tells us that in the neighborhood of the point $(0,0;\ 0;\ 0,0;\ 0,\ s_o,\ 0)$ equation (1.1) can be solved explicitly in the form

(6.12) $s = f(x,y;\ u;\ p,q;\ r,t).$

Under hypothesis 3) or 3)', the function $f \in C'''$ or C'', respectively, in a neighborhood of this point. Moreover,

(6.13) $$f_{r_o} = f_{t_o} = 0 \quad \text{and } s_o = f_o$$

while the hyperbolic condition becomes at the origin

(6.14) $$1 - 4 f_{r_o} f_{t_o} = 1 > 0$$

and the equation for the characteristic base curves becomes

(6.15) $$f_r dy^2 + dxdy + f_t dx^2 = 0.$$

Let us assume that we have a particular integral surface $J:\ u = u(x,y)$ passing through the coordinate axes in a neighborhood of the origin, with $u(x,y) \in C'''$ in this neighborhood..
We define

$$(6.16) \qquad \delta = \sqrt{1-4\,f_r f_t}\,, \quad \rho = \frac{-2f_t}{1+\delta}\,, \quad \sigma = \frac{-2f_r}{1+\delta}\,,$$

δ, ρ and σ being of class C'' by hypothesis 3), or of class C' by hypothesis 3)', in the variables x,y; u; p,q; r,t in a neighborhood of the point (0,0; 0; 0,0; 0,0). The two one-parameter families of characteristic base curves corresponding to J are thus represented by the equations

$$(6.17) \qquad y_\lambda = \rho\, x_\lambda$$

$$(6.18) \qquad x_\mu = \sigma\, y_\mu\,.$$

Note that $\delta_o = 1$, hence $\delta > 0$ in a neighborhood of the origin, while $\rho_o = \sigma_o = 0$.

As in Chapter V, to obtain the system of characteristic equations, we transform to the characteristic base curves as coordinates and consider what relations must be satisfied along these coordinates for any given integral surface J. In particular, we specialize the transformation

$$(6.19) \qquad \begin{cases} x = x(\lambda, \mu) \\ y = y(\lambda, \mu) \end{cases}$$

by stipulating that a line λ = constant shall have x-intercept $(\lambda, 0)$ and a line μ = constant shall have y-intercept $(0, \mu)$, with $\lambda = \mu = 0$ at the origin. The Jacobian of this transformation, evaluated at the origin, has the value

$$(6.20) \qquad x_{\lambda o} y_{\mu o} - y_{\lambda o} x_{\mu o} = x_{\lambda o} y_{\mu o}(1-\rho_o\sigma_o) = x_{\lambda o} y_{\mu o} \neq 0,$$

since if $x_{\lambda_o} = 0$, then $y_{\lambda_o} = 0$ by (6.17), contradicting the requirement that $\dot{x}^2 + \dot{y}^2 \neq 0$ along any characteristic curve.

Similarly, if $y_{\mu_0} = 0$, then $x_{\mu_0} = 0$ by (6.18) and the contradiction is again obtained.

Paralleling our development in Chapter V, we see that certain determinants must vanish at each point of the integral surface J, yielding equations which must be satisfied along the characteristics on J. We have

$$(6.21)\quad \begin{vmatrix} f_r & -[f]_x & f_t \\ x_\lambda & r_\lambda & 0 \\ 0 & s_\lambda & y_\lambda \end{vmatrix} = f_r r_\lambda y_\lambda + f_t s_\lambda x_\lambda + [f]_x x_\lambda y_\lambda = 0$$

where

$$(6.22)\quad [f]_x = f_p r + f_q f + f_u p + f_x .$$

also

$$(6.23)\quad \begin{vmatrix} f_r & -[f]_y & f_t \\ x_\lambda & s_\lambda & 0 \\ 0 & t_\lambda & y_\lambda \end{vmatrix} = f_r s_\lambda y_\lambda + f_t t_\lambda x_\lambda + [f]_y x_\lambda y_\lambda = 0$$

where

$$(6.24)\quad [f]_y = f_p f + f_q t + f_u q + f_y .$$

Eliminating s_λ between (6.21) and (6.23), we obtain

$$(6.25)\quad f_r^2 r_\lambda y_\lambda^2 - f_t^2 t_\lambda x_\lambda^2 + [f]_x f_r x_\lambda y_\lambda^2 -$$
$$[f]_y f_t x_\lambda^2 y_\lambda = 0.$$

By virtue of definitions (6.16) and equation (5.17), we may write (6.25) as

$$(6.26)\quad f_t^2 x_\lambda^2 \cdot H(\lambda, \mu) = 0$$

where

$$(6.27) \qquad H(\lambda,\mu) = r_\lambda \sigma^2 - t_\lambda + \frac{2}{1+\delta}\left\{[f]_y - \sigma[f]_x\right\} x_\lambda .$$

But, as shown above, $x_\lambda \neq 0$ along any of the characteristic base curves of J of the corresponding family, hence (6.26) reduces to

$$(6.28) \qquad f_t^2 \cdot H(\lambda,\mu) = 0.$$

Where $f_t \neq 0$ we have immediately that $H(\lambda,\mu) = 0$. Suppose at a particular point of J that $f_t = 0$. Then by (6.16) and (6.17), we have there that

$$(6.29) \qquad \rho = 0, \qquad \delta = 1, \qquad \sigma = -f_r \quad \text{and } y_\lambda = 0.$$

Thus, at this point, by (6.24),

$$(6.30) \qquad t_\lambda = s_y x_\lambda = (f_r r_y + [f]_y) x_\lambda ;$$

while by (6.22),

$$(6.31) \qquad r_\lambda \sigma^2 = f_r^2 r_x x_\lambda = f_r^2 (s_\lambda - [f]_x x_\lambda).$$

Substituting (6.30) and (6.31) into (6.27), we obtain that where $f_t = 0$ on J, $H(\lambda,\mu) = 0$. Hence by (6.28), $H(\lambda,\mu) = 0$ everywhere on J and represents a relation which must be satisfied along each characteristic of the corresponding family on J.

For the other family of characteristics on J, we have determinants corresponding to (6.21) and (6.22) which vanish at each point of J. Eliminating s_μ between these and arguing in a fashion analogous to that above, we arrive at the following rela-

tion which must be satisfied along each characteristic of this family on J:

$$(6.32)\quad K(\lambda,\mu) = \rho^2 t_\mu - r_\mu + \frac{2}{1+\delta}\left\{[f]_x - \rho[f]_y\right\} y_\mu = 0.$$

We are now in a position to prescribe the necessary and sufficient conditions that the coordinate axes be characteristics for any integral surface of

$$(6.12)\qquad s = f(x,y;\ u;\ p,q;\ r,t)$$

passing through them.

Suppose that, in a neighborhood of the origin, the coordinate axes are characteristic to some integral surface J: $u=u(x,y)$ of (6.12) passing through them. Then in terms of the characteristic base curves to J as coordinates, defined by the coordinate transformation (6.19), we have for $\mu = 0$:

$$x = \lambda,\ y = 0,\ u = p = r = 0,\ q = Q(\lambda),\ t = T(\lambda),$$

where, from (6.12),

$$(6.33)\quad Q'(\lambda) = f(\lambda,0;\ 0;\ 0,\ Q(\lambda);\ 0,\ T(\lambda)),$$

while, from $H(\lambda,\mu) = 0$, since $\rho = f_t = 0$, $\delta = 1$ and $\sigma = -f_r$,

$$(6.34)\quad T'(\lambda) = \left\{[f]_y + f_r[f]_x\right\}(\lambda,0;\ 0;\ 0,\ Q(\lambda);\ 0,T(\lambda)).$$

Moreover,

$$(6.35)\qquad Q(0) = T(0) = 0.$$

Equations (6.33) and (6.34) represent a system of first order ordinary differential equations under one point boundary conditions (6.35). The right hand sides of the equations of this system are of class C'' under hypothesis 3), or of class C' under hypothesis 3)', in the variables λ, Q and T. Hence, in either case, the functions Q and T are uniquely determined in a neighborhood of $\lambda = 0$. If the x axis is characteristic, these functions must also satisfy

(6.36) $$f_t(\lambda, 0; 0; 0, Q(\lambda); 0, T(\lambda)) = 0.$$

Similarly, for $\lambda = 0$:

$x = 0,\ y = \mu,\ u = q = t = 0,\ p = P(\mu),\ r = R(\mu),$

where, from (6.12),

(6.37) $$P'(\mu) = f(0, \mu; 0; P(\mu), 0; R(\mu), 0),$$

while, from $X(\lambda, \mu) = 0$, since $\sigma = f_r = 0$, $\delta = 1$ and $\rho = -f_t$,

(6.38) $$R'(\mu) = \left\{ [f]_x + f_t [f]_y \right\} (0, \mu; 0; P(\mu), 0; R(\mu), 0).$$

Moreover,

(6.39) $$P(0) = R(0) = 0.$$

Hence, if the y axis is characteristic, the functions P and R, uniquely determined by (6.37), (6.38), and (6.39), must also satisfy

(6.40) $$f_r(0, \mu; 0; P(\mu), 0; R(\mu), 0) = 0.$$

To recapitulate, the necessary condition that the x axis be a characteristic of some integral surface is that the functions Q and T determined from the system (6.33) and (6.34), under boundary conditions (6.35), shall satisfy (6.36) for each λ in a neighborhood of $\lambda = 0$. The necessary condition that the y axis be a characteristic of some integral surface is that the functions P and R determined from the system (6.37) and (6.38, under boundary conditions (6.39), shall satisfy (6.40) for each μ in a neighborhood of $\mu = 0$.

We now show that these conditions are also sufficient, i.e. given in the vicinity of the origin, an integral surface J: $u = u(x,y)$ of (6.12) passing through the coordinate axes, with

$$(6.41)\quad P_1(y) = u_x(o,y),\ R_1(y) = u_{xx}(0,y),\ Q_1(x) = u_y(x,0),\ \text{and } T_1(x) = u_{yy}(x,0),$$

we show that the requirement

$$(6.40)'\quad f_r(0,y;\ 0;\ P_1(y),0;\ R_1(y),0) = 0$$

is sufficient that the y axis be a characteristic on J.

The argument needed to show that the requirement

$$(6.36)'\quad f_t(x,0;\ 0;\ 0,Q_1(x);\ 0,\ T_1(x)) = 0$$

is sufficient in order that the x axis be a characteristic on J is analogous to the following and will not be given here.

We need show only that under requirement (6.40)', $P_1(y) = P(y)$ and $R_1(y) = R(y)$, where P(y) and R(y) are those functions obtained

previously under the assumption that the y-axis was "intrinsically characteristic".

Now $P_1(0) = R_1(0) = 0$ since $u(x,0) = 0$. Moreover, since u satisfies

$$s = f(x,y;\, u;\, p,q;\, r,t), \tag{6.12}$$

for $x = 0$,

$$P_1{}'(y) = f(0,y;\, 0;\, P_1(y),\, 0;\, R_1(y),0). \tag{6.37)'}$$

Now, recalling that $u \in C'''$,

$$s_x = f_r\, r_x + f_t\, t_x + [f]_x, \tag{6.42}$$

$$s_y = f_r\, r_y + f_t\, t_y + [f]_y. \tag{6.43}$$

Since $u(0,y) = 0$, we obtain $t_y(0,y) = 0$. Writing $r_x(0,y) = w(y)$ and substituting (6.43) into (6.42) with $x = 0$, we obtain

$$\begin{aligned} s_x(0,y) &= r_y(0,y) \\ &= f_r\, w(y) + f_t\, f_r\, r_y + [f]_x + f_t\, [f]_y \end{aligned} \tag{6.44}$$

But, $u(0,y) = u_y(0,y) = u_{yy}(0,y) = 0$, hence by (6.44),

$$R_1{}'(y) = \left[\frac{1}{1-f_r f_t}\left\{[f]_x + f_t[f]_y + f_r w(y)\right\}\right](0,y;\, 0;\, P_1(y),0;\, R_1(y),0). \tag{6.38)'}$$

Now equation (6.37)' is precisely the same as (6.37), while requirement (6.40)' is sufficient to reduce (6.38)' to (6.38). But this implies that $P_1(y) = P(y)$ and $R_1(y) = R(y)$ since the solution of the system of ordinary differential equations in question is unique.

In the foregoing arguments we have developed a procedure for determining whether or not the initial curves are "intrinsically characteristic". By transformation (6.1) and substitution (6.5), we reduce the initial curves Γ_1 and Γ_2 to the coordinate axes. If now s_0 can be uniquely determined from (6.9) we may verify the hyperbolic condition and obtain the characteristic directions at the origin. If these directions coincide with the coordinate axes, then equation (1.1) can be solved explicitly for (6.12). From this, the system (6.37) and (6.38) under boundary condition (6.39) can, in principle at least, be solved for functions P and R. Finally if P and R satisfy (6.40) then the y axis is characteristic to any integral surface of the problem, i.e. "intrinsically characteristic". Likewise, from the system (6.33) and (6.34) under boundary condition (6.35), the functions Q and T can be determined. If these satisfy (6.36) then the x axis is "intrinsically characteristic". Note that P, R, Q and T are evidently of class C'.

Having given hypothesis 2) a precise meaning along with a procedure for determining whether or not it is verified for a given problem, we continue with the proof under the assumption that hypothesis 2) is verified.

From equations (6.17), (6.18), (6.27), (6.32) and the strip conditions we obtain the following system of characteristic equations, which must be satisfied along the characteristics on any integral surface J:

$$(6.45)\qquad \left.\begin{aligned} \varphi_1 &= y_\lambda - \rho x_\lambda = 0 \\ \varphi_2 &= r_\lambda \sigma^2 - t_\lambda + \frac{2}{1+\delta}\left\{[f]_y - \sigma [f]_x\right\} x_\lambda = 0 \\ \varphi_3 &= u_\lambda - p x_\lambda - q y_\lambda = 0 \\ \varphi_4 &= p_\lambda - r x_\lambda - f y_\lambda = 0 \\ \varphi_5 &= q_\lambda - f x_\lambda - t y_\lambda = 0 \\ \psi_1 &= x_\mu - \sigma y_\mu = 0 \\ \psi_2 &= r_\mu - \rho^2 t_\mu - \frac{2}{1+\delta}\left\{[f]_x - \rho [f]_y\right\} y_\mu = 0 \end{aligned}\right\}\ \begin{matrix}\text{System}\\ \text{A}\end{matrix}$$

$$\left.\begin{aligned} \psi_3 &= u_\mu - p x_\mu - q y_\mu = 0 \\ \psi_4 &= p_\mu - r x_\mu - f y_\mu = 0 \\ \psi_5 &= q_\mu - f x_\mu - t y_\mu = 0 \end{aligned}\right\}\ \begin{matrix}\text{System}\\ \text{B}\end{matrix}$$

We observe that System A of (6.45) is of canonical hyperbolic form in x,y; u; p,q; r,t as functions of λ and μ. Since for Theorem 9, $F \in C'''$, while for Theorem 9a, $F \in C''$, the coefficients of all equations in (6.45) are functions of class C'' for Theorem 9, and of class C' for Theorem 9a. Moreover, the matrix of coefficients for System A is, after interchange of rows and columns,

$$(6.46)\qquad \begin{vmatrix} -\rho & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\sigma & 0 & 0 & 0 & 0 & 0 \\ * & 0 & \sigma^2 & -1 & 0 & 0 & 0 \\ 0 & * & 1 & -\rho^2 & 0 & 0 & 0 \\ * & * & 0 & 0 & 1 & 0 & 0 \\ * & * & 0 & 0 & 0 & 1 & 0 \\ * & * & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (1-\rho\sigma)(\rho^2\sigma^2-1) = \frac{-8\delta^2}{(1+\delta)^3}$$

where the coefficients designated only by asterisks, *, do not contribute to the value of the determinant. But $\delta > 0$ everywhere on J in a neighborhood of the origin, hence the determinant (6.46) does not vanish thereon.

As to the initial conditions, we have, by hypothesis 1) of Theorems 9 and 9a for $\mu = 0$,

$$x = \lambda , \; y = 0, \quad u = p = r = 0, \; q = Q(\lambda), \; t = T(\lambda),$$

and for $\lambda = 0$,

$$x = 0, \; y = \mu , \; u = q = t = 0, \quad p = P(\mu), \; r = R(\mu)$$

where Q, T and P,R are determined from their respective systems and are of class C'. Moreover, for $\mu = 0$, by (6.36), $f_t = 0$. Hence $\rho = 0$, $\delta = 1$, and $\sigma = - f_r$. This together with $y_\lambda = r_\lambda = u_\lambda = p_\lambda = 0$ and equation (6.34) prove that

$$(6.47) \quad \varphi_1(\lambda,0) = \varphi_2(\lambda,0) = \varphi_3(\lambda,0) = \varphi_4(\lambda,0) = \varphi_5(\lambda,0) = 0$$

for all λ in a neighborhood of $\lambda = 0$. Similarly, for $\lambda = 0$, by (6.40), $f_r = 0$. Hence $\sigma = 0$, $\delta = 1$ and $\rho = - f_t$. This together with $x_\mu = t_\mu = u_\mu = q_\mu = 0$ and equation (6.38) prove that

$$(6.48) \quad \psi_1(0,\mu) = \psi_2(0,\mu) = \psi_3(0,\mu) = \psi_4(0,\mu) = \psi_5(0,\mu) = 0$$

for all μ in a neighborhood of $\mu = 0$. Thus the initial condition requirements of hypothesis 3) of Theorems 6 and 6a are satisfied.

Since the coefficients in (6.45) are of class C'' for Theorem 9, hypotheses 1) and 2) of Theorem 6 are satisfied. Also, since the coefficients in (6.45) are of class C' for Theorem 9a, the

common hypothesis 1) of Theorems 6 and 6a is satisfied, but hypothesis 2) of Theorem 6, a hypothesis which does not appear in Theorem 6a, is not satisfied. Thus if we now show that any solution of the canonical hyperbolic system, System A of (6.45), with the given characteristic initial conditions is also a solution of the corresponding problem for the equation

$$(6.12) \qquad s = f(x,y;\ u;\ p,q;\ r,t)$$

with the same initial conditions, then Theorem 9 is an immediate consequence of Theorem 6 and Theorem 9a is an immediate consequence of Theorem 6a.

As in the Cauchy problem of Chapter 5, we show that for each solution of System A under the given characteristic initial conditions that System B is likewise satisfied. Note that here we cannot assume that p,q,r and t are derivatives of u; this is a matter of proof. Recalling from Theorems 6 and 6a that the functions of the solution of System A, x,y,u,p,q,r,t are of class C' and that $f \in C'''$ under hypothesis 3) of Theorem 9, or $f \in C''$ under hypothesis 3)' of Theorem 9a, we obtain by differentiation and consideration of (6.45) that

$$(6.49) \qquad \begin{aligned} \psi_{3,\lambda} - \varphi_{3,\mu} &= p_\mu x_\lambda + q_\mu y_\lambda - p_\lambda x_\mu - q_\lambda y_\mu \\ &= \psi_4 x_\lambda + \psi_5 y_\lambda - \varphi_4 x_\mu - \varphi_5 y_\mu . \end{aligned}$$

Moreover, since $\varphi_3 = \varphi_4 = \varphi_5 = 0$,

$$(6.50) \qquad \begin{aligned} f_\lambda &= f_r r_\lambda + f_t t_\lambda + f_p p_\lambda + f_q q_\lambda + f_u u_\lambda + f_x x_\lambda + f_y y_\lambda \\ &= f_r r_\lambda + f_t t_\lambda + [f]_x x_\lambda + [f]_y y_\lambda , \end{aligned}$$

while

$$(6.51)\quad \begin{aligned} f_\mu &= f_r r_\mu + f_t t_\mu + f_p p_\mu + f_q q_\mu + f_u u_\mu + f_x x_\mu + f_y y_\mu \\ &= f_r r_\mu + f_t t_\mu + [f]_x x_\mu + [f]_y y_\mu \\ &\qquad + f_p \psi_4 + f_q \psi_5 + f_u \psi_3 . \end{aligned}$$

Thus by (6.45), (6.50) and (6.51),

$$(6.52)\quad \begin{aligned} \psi_{4,\lambda} - \varphi_{4,\mu} &= r_\mu x_\lambda + f_\mu y_\lambda - r_\lambda x_\mu - f_\lambda y_\mu \\ &= y_\lambda \left\{ f_p \psi_4 + f_q \psi_5 + f_u \psi_3 \right\} \\ &\quad + \left(\frac{1+\delta}{2}\right) x_\lambda \psi_2 - \left(\frac{1+\delta}{2}\right) \rho y_\mu \varphi_2 , \end{aligned}$$

and

$$(6.53)\quad \begin{aligned} \psi_{5,\lambda} - \varphi_{5,\mu} &= f_\mu x_\lambda + t_\mu y_\lambda - f_\lambda x_\mu - t_\lambda y_\mu \\ &= x_\lambda \left\{ f_p \psi_4 + f_q \psi_5 + f_u \psi_3 \right\} \\ &\quad - \left(\frac{1+\delta}{2}\right) \sigma x_\lambda \psi_2 + \left(\frac{1+\delta}{2}\right) y_\mu \varphi_2 . \end{aligned}$$

Taking into account the fact that System A is satisfied, we reduce (6.49), (6.52) and (6.53) to the system

$$(6.54)\quad \begin{aligned} \psi_{3,\lambda} &= \psi_4 x_\lambda + \psi_5 y_\lambda \\ \psi_{4,\lambda} &= y_\lambda \left\{ f_u \psi_3 + f_p \psi_4 + f_q \psi_5 \right\} \\ \psi_{5,\lambda} &= x_\lambda \left\{ f_u \psi_3 + f_p \psi_4 + f_q \psi_5 \right\} \end{aligned}$$

For fixed μ, (6.54) represents a system of linear, homogeneous, first order ordinary differential equations for the functions ψ_3, ψ_4 and ψ_5 of the variable λ. Moreover, by (6.48),

the homogeneous one point boundary conditions

$$\psi_3(0,\mu) = \psi_4(0,\mu) = \psi_5(0,\mu) = 0$$

must be satisfied. Hence, the unique solution for the system (6.54) is

$$\psi_3 = \psi_4 = \psi_5 = 0$$

wherever the solution of system A is defined.

Consider the linear algebraic system,

$$(6.55)\quad \begin{cases} \varphi_3 = u_\lambda - px_\lambda - qy_\lambda = 0 \\ \psi_3 = u_\mu - px_\mu - qy_\mu = 0 . \end{cases}$$

The determinant of this system, by (6.20), does not vanish in a neighborhood of the origin, hence in this neighborhood there exists a unique solution for p and q. Since $p = u_x$ and $q = u_y$ satisfy (6.55) they are the solution of (6.55)

Similarly, from

$$(6.56)\quad \begin{cases} \varphi_4 = p_\lambda - r\,x_\lambda - fy_\lambda \\ \psi_4 = p_\mu - r\,x_\mu - fy_\mu , \end{cases}$$

we obtain $r = u_{xx}$ and $f = u_{xy}$,
while from

$$(6.57)\quad \begin{cases} \varphi_5 = q_\lambda - fx_\lambda - ty_\lambda \\ \psi_5 = q_\mu - fx_\mu - ty_\mu , \end{cases}$$

we obtain the additional information that $t = u_{yy}$. Consequently, any solution of System A under the given characteristic initial conditions satisfies the equation

$$u_{xy} = f(x,y;\, u;\, u_x,\, u_y;\, u_{xx},\, u_{yy})$$

in a neighborhood of the point (0,0; 0; 0,0; 0,0) and the proof of Theorems 9 and 9a is now complete.

Let us designate the problem considered in Theorems 9 and 9a as Problem I. By virtue of the exposition of Chapter IV and this present chapter, we may associate to this problem a particular Problem II, of the type considered in Theorems 3 and 3a of Chapter II. As we have shown, any solution of I is a solution of II, and, conversely, any solution of II is a solution of I. Where for I, $F \in C'''$, Theorem 3 tells us that the solution of the related Problem II is unique. Hence, as is stated in Theorem 9, the solution for I is likewise unique. If, however, for Problem I, $F \in C''$ only, then Theorem 3a tells us merely that the related Problem II has at least one solution. Moreover, Example 1, Chapter II, tells us that this solution cannot be shown to be unique.

We must not conclude merely from the above that for $F \in C''$ the solution to Problem I cannot be shown to be unique. We can say, though, that any proof for uniqueness, if such can be made at all, will apparently have to be based upon arguments independent of those of this paper.

Chapter VII

The Mixed Boundary Value Problem for $u_{xy} = f(x,y;\, u;\, u_x,\, u_y)$.

In the terminology of J. HADAMARD [11], appendix II, p. 456, the mixed hyperbolic boundary value problem is one in which we prescribe the values of the integral surface along two lines issuing from a point, one of which is characteristic to the surface in question, while the other is nowhere characteristic.

J. HADAMARD, in the reference above, and E. PICARD [7], p.135, prove the existence of a unique solution to the linear equation

$$(7.1) \qquad u_{xy} = a\,u_x + b\,u_y + c\,u\ ,$$

a, b and c continuous functions of x and y alone, satisfying the initial conditions

$$(7.2) \qquad u(x,0) = u(x,x) = 0.$$

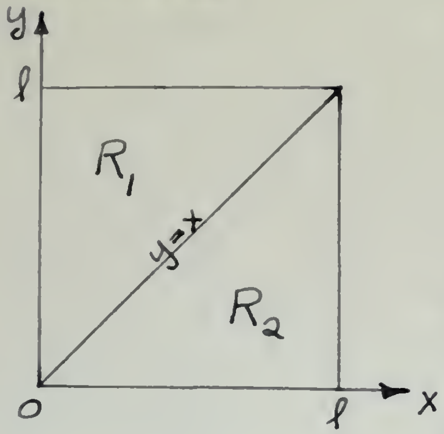
In Theorem 10, below, we extend their conclusions to the equation

$$(7.3) \qquad u_{xy} = f(x,y;\, u;\, u_x,u_y)$$

maintaining initial conditions (7.2). The result is well known, but does not appear in the literature in the precise form stated. We require this precise statement because we wish to proceed from Theorem 10 by the methods of Chapters II and III in which we relax the Lipschitz condition on the function f to require merely

that f be partially Lipschitzian. Thus we obtain the improved statement of Theorem 10a.

Theorem 10



1) $f(x,y;\, u;\, p,q) \in C(B)$, $B: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \\ -a \leq u \leq a \\ -b \leq p \leq b \\ -b \leq q \leq b \end{cases}$

2) f is Lipschitzian on B (as defined in Theorem 1.)

3) $M\ell^2 \leq a$, $M\ell \leq b$, where

$M = \max |f|$ on B

4) There exists one and only one function $u(x,y) \in C'(R)$, $u_{xy}(x,y) \in C(R)$, where $R: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \end{cases}$, such that for each $(x,y) \in R$, the point $(x,y;\, u(x,y);\, u_x(x,y),\, u_y(x,y)) \in B$, and

$$u_{xy}(x,y) = f(x,y;\, u(x,y);\, u_x(x,y),\, u_y(x,y)),$$

$$u(x,0) = u(x,x) = 0 \quad \text{for each } (x,y) \in R.$$

Proof

This proof is based upon PICARD's variation of the method of successive approximations, [1] p. 359 or [7] p. 117. Here the uniform convergence of the approximating functions to the solution is verified by means of a majorant series. The majorant series used is that obtained from the approximating functions converging uniformly to the solution for the particular linear equation

$$(7.4) \qquad W_{xy} = K\,(W + W_x + W_y)$$

with the same initial conditions. K is the Lipschitz constant for the function f of (7.3). PICARD applied this technique to the characteristic initial value problem, obtaining Theorem 1 of Chapter II. He thus obtained the theorem for the characteristic initial value problem for the non-linear equation (7.3) from the theorem for the characteristic initial value problem for the linear equation (7.1).

For the mixed boundary value problem under consideration, a curious situation arises. We do not obtain a majorant series from equation (7.4) under mixed initial conditions. However, we do find that PICARD's majorant series for the characteristic initial value problem serves as well for this problem. Thus Theorem 10 follows not from the theorem for the mixed boundary value problem for the linear equation (7.1) but from the theorem for the characteristic initial value problem for equation (7.1).

It is sufficient, as we shall demonstrate later, to show existence of a unique solution in region $R_2: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq x \end{cases}$. Assuming $(x,y) \in R_2$, we may express the problem as the integral equation

$$(7.5) \qquad u(x,y) = \int_y^x d\xi \int_0^y f(\xi, \eta\, ;\, u;\, u_x, u_y)\, d\eta \,.$$

By differentiation,

$$(7.6) \qquad u_x(x,y) = \int_0^y f(x, \eta\, ;\, u;\, u_x, u_y)\, d\eta \,,$$

and

$$(7.7)\quad u_y(x,y) = \int_y^x f(\xi, y; u; u_x, u_y)\,d\xi - \int_0^y f(y, \eta; u; u_x, u_y)\,d\eta.$$

We form the successive approximations

$$(7.8)\quad \begin{cases} u_1(x,y) = \int_y^x d\xi \int_0^y f(\xi, \eta; 0; 0,0)\,d\eta \\ u_2(x,y) = \int_y^x d\xi \int_0^y f(\xi, \eta; u_1; u_{1,x}, u_{1,y})\,d\eta \\ \quad\vdots \qquad\qquad\qquad \vdots \\ u_n(x,y) = \int_y^x d\xi \int_0^y f(\xi, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y})\,d\eta \\ \quad\vdots \qquad\qquad\qquad \vdots \end{cases}$$

where, by differentiation,

$$(7.9)\quad u_{n,x}(x,y) = \int_0^y f(x, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y})\,d\eta, \quad (n = 1,2,\cdots),$$

$$(7.10)\quad u_{n,y}(x,y) = \int_y^x f(\xi, y; u_{n-1}; u_{n-1,x}, u_{n-1,y})\,d\xi - \int_0^y f(y, \eta; u_{n-1}; u_{n-1,x}, u_{n-1,y})\,d\eta, \quad (n = 1,2,\cdots,).$$

Since the point $(x,y; 0; 0,0) \in B$ for $(x,y) \in R_2$, by hypothesis 3),

$$\begin{aligned} |u_1(x,y)| &\le M\,|x-y| \cdot |y| \le M\ell^2 \le a, \\ |u_{1,x}(x,y)| &\le M\,|y| \le M\ell \le b, \\ |u_{1,y}(x,y)| &\le M\left\{|x-y| + |y|\right\} \\ &= M|x| \le M\ell \le b \end{aligned}$$

Thus, by induction, for all n and for any $(x,y) \in R_2$

$$(7.11)\quad \begin{cases} |u_n(x,y)| \le M\ell^2 \le a, \\ |u_{n,x}(x,y)| \le M\ell \le b, \\ |u_{n,y}(x,y)| \le M\ell \le b. \end{cases}$$

Our purpose is to show that on R_2

$$(7.12)\quad \{u_n\} \xrightarrow{\text{unif}} u, \ \{u_{n,x}\} \xrightarrow{\text{unif}} u_x \text{ and } \{u_{n,y}\} \xrightarrow{\text{unif}} u_y$$

such that the function u and its derivatives satisfy conclusion 4) for $(x,y) \in R_2$. To accomplish this we consider the successive approximations

$$(7.13)\quad \begin{aligned} w_1(x,y) &= \int_0^x d\xi \int_0^y M\, d\eta \\ w_2(x,y) &= \int_0^x d\xi \int_0^y K(w_1 + w_{1,x} + w_{1,y})d\eta \\ &\vdots \\ w_n(x,y) &= \int_0^x d\xi \int_0^y K(w_{n-1} + w_{n-1,x} + w_{n-1,y})d\eta \\ &\vdots \end{aligned}$$

where, by differentiation,

$$(7.14)\quad w_{n,x}(x,y) = \int_0^y K\left[w_{n-1} + w_{n-1,x} + w_{n-1,y}\right](x,\eta)d\eta\ , \quad (n = 1,2,\cdots),$$

$$(7.15)\quad w_{n,y}(x,y) = \int_0^x K\left[w_{n-1} + w_{n-1,x} + w_{n-1,y}\right](\xi,y)d\xi\ , \quad (n = 1,2,\cdots).$$

Here $M = \max |f|$ on R while K is the Lipschitz constant of hypothesis 2).

Now $w_1(x,y) = Mxy$, hence $w_1(x,y) = w_1(y,x)$. Moreover, $w_{1,x}(x,y) = My$, $w_{1,y}(x,y) = Mx$, hence $w_{1,x}(x,y) = w_{1,y}(y,x)$.

Let us make the inductive hypothesis that for some fixed positive integer n,

$$(7.16)\quad w_n(x,y) = w_n(y,x),\ w_{n,x}(x,y) = w_{n,y}(y,x).$$

But this implies that

$$(7.17)\quad \left[w_n + w_{n,x} + w_{n,y}\right](x,y) = \left[w_n + w_{n,x} + w_{n,y}\right](y,x)$$

and thus, by (7.13),

$$w_{n+1}(x,y) = w_{n+1}(y,x).$$

Also, by (7.14) and (7.15), (7.17) implies that

$$\begin{aligned} w_{n+1,x}(x,y) &= \int_0^y K\left[w_n + w_{n,x} + w_{n,y}\right](x,\eta)\,d\eta \\ &= \int_0^y K\left[w_n + w_{n,x} + w_{n,y}\right](\xi,x)\,d\xi \\ &= w_{n+1,y}(y,x). \end{aligned}$$

Hence, by induction, (7.16) holds for $n = 1,2,\cdots$.

PICARD, in the references quoted above, shows that

$$(7.18)\quad \sum_{n=1}^{\infty} w_n = w, \quad \sum_{n=1}^{\infty} w_{n,x} = w_x, \quad \sum_{n=1}^{\infty} w_{n,y} = w_y,$$

each uniformly convergent on R, where the function w and its derivatives satisfy

$$(7.19)\quad \begin{aligned} &w_{xy} = K(w + w_x + w_y), \\ &w(x,0) = w(0,y) = 0. \end{aligned}$$

We now show that these series are majorant to the series

$$(7.20)\quad \sum_{n=1}^{\infty} (u_n - u_{n-1}), \quad \sum_{n=1}^{\infty} (u_{n,x} - u_{n-1,x}), \quad \sum_{n=1}^{\infty} (u_{n,y} - u_{n-1,y}),$$

respectively, for each $(x,y) \in R_2$, (with $u_0 = 0$).

Now, for $(x,y) \in R_2$,

$$|u_1(x,y)| \leq \int_y^x d\xi \int_0^y |f(\xi,\eta\,;\,0;\,0,0)|\,d\eta \leq \int_0^x d\xi \int_0^y M\,d\eta = w_1(x,y)$$

$$|u_{1,x}(x,y)| \leq \int_0^y |f(x,\eta\,;0;0,0)|\,d\eta \leq \int_0^y M\,d\eta = w_{1,x}(x,y)$$

$$
\begin{aligned}
|u_{1,y}(x,y)| &\leq \int_y^x |f(\xi,y;0;0,0)|\,d\xi + \int_0^y |f(y,\eta;0;0,0)|\,d\eta \\
&\leq \int_y^x M d\xi + \int_0^y M d\eta \\
&= \int_0^x M d\xi = w_{1,y}(x,y).
\end{aligned}
$$

Also, abbreviating our notation somewhat,

$$
\begin{aligned}
|u_2-u_1| &\leq \int_y^x d\xi \int_0^y |f(\xi,\eta;u_1;u_{1,x},u_{1,y}) \\
&\qquad -f(\xi,\eta;0;0,0)|\,d\eta \\
&\leq \int_y^x d\xi \int_0^y K\left[|u_1| + |u_{1,x}| + |u_{1,y}|\right](\xi,\eta)d\eta \\
&\leq \int_0^x d\xi \int_0^y K\left[w_1 + w_{1,x} + w_{1,y}\right](\xi,\eta)d\eta \\
&= w_2,
\end{aligned}
$$

$$
|u_{2,x}-u_{1,x}| \leq \int_0^y K\left[w_1 + w_{1,x} + w_{1,y}\right](x,\eta)d\eta = w_{2,x}
$$

$$
\begin{aligned}
|u_{2,y}-u_{1,y}| &\leq \int_y^x K\left[w_1 + w_{1,x} + w_{1,y}\right](\xi,y)d\xi \\
&\quad + \int_0^y K\left[w_1 + w_{1,x} + w_{1,y}\right](y,\eta)d\eta \\
&= \int_y^x K\left[w_1 + w_{1,x} + w_{1,y}\right](\xi,y)\,d\xi \\
&\quad + \int_0^y K\left[w_1 + w_{1,x} + w_{1,y}\right](\xi,y)d\xi \\
&= \int_0^x K\left[w_1 + w_{1,x} + w_{1,y}\right](\xi,y)d\xi \\
&= w_{2,y}.
\end{aligned}
$$

Hence, by induction, we obtain for $n = 1,2,\cdots$

$$
|u_n - u_{n-1}| \leq w_n, \quad |u_{n,x} - u_{n-1,x}| \leq w_{n,x},
$$

(7.21) $$|u_{n,y} - u_{n-1,y}| \leq w_{n,y} \quad \text{for each } (x,y) \in R_2.$$

Thus the series of (7.18) are majorant to the corresponding series of (7.20). Moreover, the requirements for termwise differentiation of an infinite sum are satisfied since each of the series of (7.20) is now known to be uniformly convergent on R_2. Hence, for $(x,y) \in R_2$,

$$(7.22)\quad \begin{cases} \sum_{n=1}^{\infty} (u_n - u_{n-1}) = u \\ \sum_{n=1}^{\infty} (u_{n,x} - u_{n-1,x}) = u_x \\ \sum_{n=1}^{\infty} (u_{n,y} - u_{n-1,y}) = u_y \, ; \end{cases}$$

or, in other terms, since each of these series telescopes,

$$(7.22)' \quad \{u_n\} \xrightarrow{\text{unif}} u, \quad \{u_{n,x}\} \xrightarrow{\text{unif}} u_x, \quad \{u_{n,y}\} \xrightarrow{\text{unif}} u_y$$

on R_2.

We now verify that the function u and its derivatives u_x and u_y satisfy the integral equation statement of the problem (7.5):

$$\begin{aligned} &\left| u(x,y) - \int_y^x d\xi \int_0^y f(\xi,\eta\,;u;\,u_x,u_y)\,d\eta \right| \\ &\quad \le \left| u(x,y) - u_n(x,y) \right| + \int_y^x d\xi \int_0^y \Big| f(\xi,\eta\,;u\,;u_x,u_y) \\ (7.23) \qquad &\qquad\qquad - f(\xi,\eta\,;u_{n-1}\,;u_{n-1,x},u_{n-1,y}) \Big|\, d\eta \\ &\quad \le \left| u(x,y) - u_n(x,y) \right| \\ &\qquad + \int_y^x d\xi \int_0^y K \Big[|u - u_{n-1}| + |u_x - u_{n-1,x}| + |u_y - \\ &\qquad\qquad u_{n-1,y}| \Big] (\xi,\eta)\, d\eta \end{aligned}$$

Thus, by (7.22)', given $\epsilon > 0$, there exists a positive integer N, depending on ϵ alone, such that $n > N \Rightarrow$

$$\left| u(x,y) - \int_y^x d\xi \int_0^y f(\xi, \eta ; u; u_x, u_y) d\eta \right| < \epsilon (1 + 3K\ell^2),$$

for $(x,y) \in R_2$. But ϵ is arbitrary, hence the integral equation is satisfied.

By (7.11) and (7.22)' we see that for any $(x,y) \in R_2$, the point $(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)) \in B$. Thus existence of a solution on R_2 is now proved.

To prove uniqueness, let us suppose that u_1 and u_2 are two solutions on R_2, then

$$\begin{aligned} |u_1(x,y) - u_2(x,y)| \leq \int_y^x d\xi \int_0^y | & f(\xi, \eta ; u_1; u_{1,x}, u_{1,y}) \\ & - f(\xi, \eta ; u_2; u_{2,x}, u_{2,y}) | d\eta \end{aligned} \tag{7.24}$$

$$\leq \int_y^x d\xi \int_0^y K[|u_1 - u_2| + |u_{1,x} - u_{2,x}| + |u_{1,y} - u_{2,y}|] (\xi, \eta) d\eta.$$

$$\begin{aligned} |u_{1,x}(x,y) - u_{2,x}(x,y)| \leq \int_0^y | & f(x, \eta ; u_1; u_{1,x}, u_{1,y}) \\ & - f(x, \eta ; u_2; u_{2,x}, u_{2,y}) | d\eta \end{aligned} \tag{7.25}$$

$$\leq \int_0^y K[|u_1 - u_2| + |u_{1,x} - u_{2,x}| + |u_{1,y} - u_{2,y}|](x, \eta) d\eta,$$

$$\begin{aligned} |u_{1,y}(x,y) - u_{2,y}(x,y)| \leq \int_y^x | & f(\xi, y; u_1; u_{1,x}, u_{1,y}) \\ & - f(\xi, y; u_2; u_{2,x}, u_{2,y}) | d\xi \end{aligned} \tag{7.26}$$

$$\begin{aligned} + \int_0^y | & f(y, \eta ; u_1; u_{1,x}, u_{1,y}) \\ & - f(y, \eta ; u_2; u_{2,x}, u_{2,y}) | d\eta . \end{aligned}$$

Let $\psi(x,y) = [\,|u_1-u_2| + |u_{1,x}-u_{2,x}| + |u_{1,y}-u_{2,y}|\,](x,y)$.

With $R^*: \begin{cases} 0 \le x \le \ell^* \\ 0 \le y \le x \end{cases}$, $\ell^* = \min(1, \ell, \frac{1}{6K})$, we have

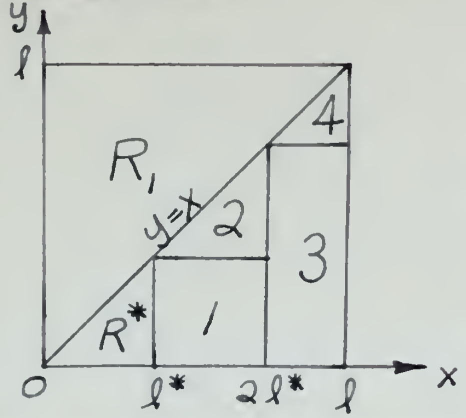
$\psi(x,y) \in C(R^*)$. Moreover, there exists a point $(x^*,y^*) \in R^*$ such that $\psi(x^*,y^*) = \mu$ where $\mu = \max \psi(x,y)$ on R^*. But, adding (7.24), (7.25) and (7.26) we obtain

$$\begin{aligned} \psi(x,y) &\le K\mu \left\{ (x-y)y + y + (x-y) + y \right\} \\ &\le K\,\mu\,(xy + x + y) \\ &\le K\,\mu \cdot \frac{3}{6K} = \frac{\mu}{2}, \end{aligned}$$

hence $\psi(x^*,y^*) = \mu \le \frac{\mu}{2}$, which implies $\mu = 0$ and thus

$$u_1(x,y) = u_2(x,y) \tag{7.27}$$

for $(x,y) \in R^*$



To extend this uniqueness proof to the domain R_2, we subdivide R_2 as shown in the diagram. We know that the solution u is unique on R^* and hence determines $u(\ell^*,y)$ for $0 \le y \le \ell^*$.

But $u(x,0) = 0$ by hypothesis, consequently, by Theorem 1, Chapter II, we have a unique solution u_1 to the characteristic initial value problem on sub-region 1. Since $u_x(\ell^*,0) = u_{1,x}(\ell^*,0)$, we have from the differential equation that $u_x(\ell^*,y) = u_{1,x}(\ell^*,y)$ for $0 \le y \le \ell^*$, i.e. u and u_1 have a first order contact across the line $x = \ell^*$ and hence together represent a unique solution for the region $R^* + 1$. Analogously, by the preceding "in the

small" uniqueness proof for the mixed boundary value problem, the solution u_2 is unique in sub-region 2 and has a first order contact with u_1 across the line $y = \ell^*$. We continue obtaining unique solutions for characteristic initial value and mixed initial value problems, alternatively as indicated by the numerical sequence in the diagram. These solutions have first order contacts with each other across the characteristics forming the boundaries of the sub-regions, hence we have extended our uniqueness proof from the region R^* to the region R_2.

Having thus determined the existence of a unique solution satisfying conclusion 4) throughout R_2, we now consider the Cauchy problem for region R_1 with the same equation and hypotheses thereon and with the initial conditions

$$(7.28)\begin{cases} u^0(x,x) = 0,\ u^0_x(x,x) = u_{x+}(x,x), \text{ and} \\ u^0_y(x,x) = u_{y-}(x,x) \quad \text{for } x \in [0,\ell]. \end{cases}$$

In (7.28) u_{x+} and u_{y-} are the right-hand x and lower y derivatives, respectively, determined at each point of the line $y = x$ by the known solution u on R_2. By Theorem 4, Chapter III, there exists a unique solution u^0 to this Cauchy problem for each $(x,y) \in R_1$, hence

$$u_1(x,y) = \begin{cases} u_0(x,y) \text{ for } (x,y) \in R_1 \\ u(x,y) \text{ for } (x,y) \in R_2 \end{cases}$$

is the unique solution valid for each $(x,y) \in R = R_1 + R_2$, since u_0 and u have, by prescription, a first order contact across the line $y = x$. This completes the proof of Theorem 10.

Relaxing only hypothesis 2) of Theorem 10, we obtain the following improvement:

Theorem 10a

1)

2)' f is partially Lipschitzian on B (as defined in Theorem 1a.)

3)

$\Longrightarrow$ 4)' There exists at least one function, etc. (as in Theorem 10.)

Outline of the proof:

As in the proof of Theorem 10, we may, without loss, prove existence on R_2 only. For, prescribing Cauchy conditions on $y = x$ as before, we may extend the solution from R_2 to R_1, by use of Theorem 4a, Chapter III.

In this proof we follow very closely the derivation of Theorem 1a, Chapter II; hence only the differences between the two proofs will be noted.

WEIERSTRASS' theorem tells us that there exists a sequence of polynomials, $\{g_\lambda\}$, converging uniformly to f on B. We extend the g_λ, ($\lambda = 1,2,\cdots$), and f from B to

$$B': \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \\ -\infty < u < \infty \\ -\infty < p < \infty \\ -\infty < q < \infty \end{cases}$$

by definitions analogous to (2.1). There exists a constant $L > 0$ such that $|g_\lambda| \leq L$ in B' and for all λ. More-

over, the g_λ are "fully" Lipschitzian in B'. Hence by Theorem 10, (with $a \to \infty$, $b \to \infty$), for each g_λ there exists a unique function u_λ such that for $(x,y) \in R_2$

$$(7.29) \quad u_\lambda = \int_y^x d\xi \int_0^y g_\lambda(\xi, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\eta ,$$

and thus

$$(7.30) \quad u_{\lambda,x} = \int_0^y g_\lambda(x, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\eta ,$$

$$(7.31) \quad u_{\lambda,y} = \int_y^x g_\lambda(\xi, y; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\xi - \int_0^y g_\lambda(y, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y}) d\eta .$$

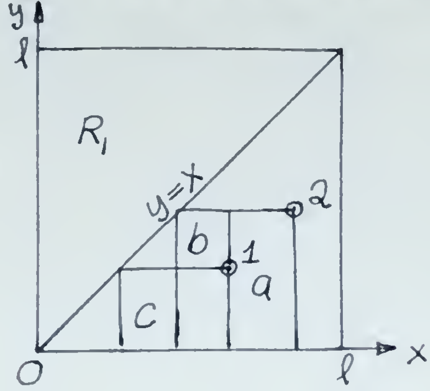
For $(x,y) \in R_2$, by (7.29), (7.30) and (7.31),

$$(7.32) \quad \left.\begin{aligned} |u_\lambda(x,y)| &\le L\,\ell^2 \\ |u_{\lambda,x}(x,y)| &\le L\,\ell \\ |u_{\lambda,y}(x,y)| &\le L\{(x-y) + y\} \\ &\le L\ell \end{aligned}\right\} (\lambda = 1, 2, \cdots)$$

i.e. the sequences $\{u_\lambda\}$, $\{u_{\lambda,x}\}$ and $\{u_{\lambda,y}\}$ are uniformly bounded on R_2.

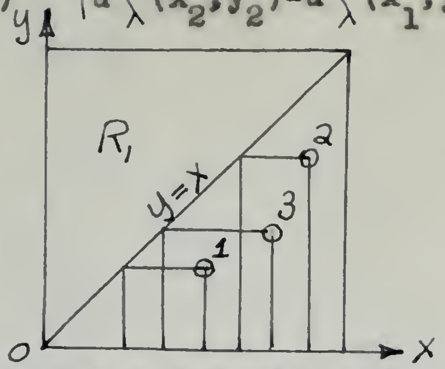
Given two points, $(x_1, y_1) \in R_2$, $(x_2, y_2) \in R_2$, we may assume, without loss, that $x_1 \le x_2$. Then, if $y_1 \le y_2$, let us assume that $y_2 < x_1$. Then by integrating over the regions a, b and c in diagram (A) we obtain

(A)



(7.33) $\left|u_\lambda(x_2,y_2)-u_\lambda(x_1,y_1)\right| \leq L\left\{\ell(x_2-x_1) + 2\ell(y_2-y_1)\right\}$.

(B)



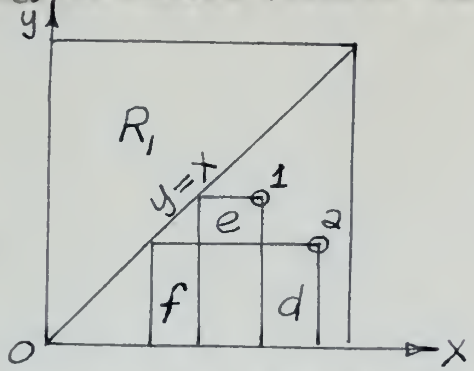
If $y_2 \geqslant x_1$ we may always choose a point (x_3,y_3) with $y_2 < x_3 < x_2$ and $y_1 < y_3 < x_1$ (as in diagram(B). Then, as above,

$$\left|u_\lambda(x_2,y_2)-u_\lambda(x_3,y_3)\right| \leq L\left\{\ell(x_2-x_3) + 2\ell(y_2-y_3)\right\}$$

$$\left|u_\lambda(x_3,y_3)-u_\lambda(x_1,y_1)\right| \leq L\left\{\ell(x_3-x_1) + 2\ell(y_3-y_1)\right\}.$$

Adding, we obtain (7.33). Further if $y_1 \geqslant y_2$, we have the case shown in diagram (C). Here by integrating over the regions d, e and f we again obtain (7.33). Hence the sequence $\{u_\lambda\}$ is equicontinuous on R_2.

(C)



Now, for $(x,y_2) \in R_2$, $(x,y_1) \in R_2$, by (7.30)

(7.34) $\left|u_{\lambda,x}(x,y_2)-u_{\lambda,x}(x,y_1)\right| \leq L\left|y_2-y_1\right|$.

Likewise, for $(x_2,y) \in R_2$, $(x_1,y) \in R_2$, by (7.31)

(7.35) $\left|u_{\lambda,y}(x_2,y) - u_{\lambda,y}(x_1,y)\right| \leq L\left|x_2 - x_1\right|$.

Moreover, by precisely the same argument as that used to prove Lemma 2 of Chapter II, given $\mu > 0$, $\zeta > 0$, there exist $\delta > 0$, $N > 0$, depending only on μ and ζ, respectively, such that for $(x_2,y) \in R_2$, $(x_1,y) \in R_2$,

$$\lambda > N \text{ and } |x_2 - x_1| < \delta$$

$\Longrightarrow$

$$(7.36)\quad |u_{\lambda,x}(x_2,y) - u_{\lambda,x}(x_1,y)| \leq K\int_0^y |u_{\lambda,x}(x_2,\eta) - u_{\lambda,x}(x_1,\eta)|d\eta + \mu + \zeta .$$

Thus by (.734), (7.36) and Lemma 1, Chapter II, the sequence $\{u_{\lambda,x}\}$ is equicontinuous on R_2.

We need the following refinement of the argument in order to show that the sequence $\{u_{\lambda,y}\}$ is equicontinuous on R_2:

Let us suppose $(x,y_2) \in R_2$, $(x,y_1) \in R_2$. Without loss, we may assume that $x \geqslant y_2 \geqslant y_1$. Then

$$\begin{aligned} & u_{\lambda,y}(x,y_2) - u_{\lambda,y}(x,y_1) \\ &= \int_{y_2}^{x} [g_\lambda(\xi,y_2;u_\lambda;u_{\lambda,x},u_{\lambda,y}) - g_\lambda(\xi,y_1;u_\lambda;u_{\lambda,x},u_{\lambda,y})]d\xi \\ (7.37)\quad & -\int_{y_1}^{y_2} g_\lambda(\xi,y_1;u_\lambda;u_{\lambda,x},u_{\lambda,y})d\xi \\ & -\int_0^{y_1}[g_\lambda(y_2,\eta;u_\lambda;u_{\lambda,x},u_{\lambda,y}) - g_\lambda(y_1,\eta;u_\lambda;u_{\lambda,x},u_{\lambda,y})]d\eta \\ & -\int_{y_1}^{y_2} g_\lambda(y_2,\eta;u_\lambda;u_{\lambda,x},u_{\lambda,y})d\eta \end{aligned}$$

We have just proved that the sequences $\{u_\lambda\}$ and $\{u_{\lambda,x}\}$ are equicontinuous on R_2. The sequence $\{g_\lambda\}$ is certainly equicontinuous on B'. Hence, considering (7.35), given $\mu > 0$, there exists $\delta > 0$, depending upon μ alone, such that $|y_2 - y_1| < \delta$

$\Longrightarrow$

$$(7.38)\quad \left|\int_0^{y_1}[g_\lambda(y_2,\eta;u_\lambda;u_{\lambda,x},u_{\lambda,y}) - g_\lambda(y_1,\eta;u_\lambda;u_{\lambda,x},u_{\lambda,y})]d\eta\right| < \mu ,$$

$$\begin{aligned} (7.39)\quad \Big|\int_{y_2}^{x} & [g_\lambda(\xi,y_2;u_\lambda(\xi,y_2);u_{\lambda,x}(\xi,y_2),\ \underline{u_{\lambda,y}(\xi,y_2)}) \\ & -g_\lambda(\xi,y_1;u_\lambda(\xi,y_1);u_{\lambda,x}(\xi,y_1),\underline{u_{\lambda,y}(\xi,y_2)})]d\xi\Big| < \mu , \end{aligned}$$

for $\lambda = 1,2,\cdots$.

Also, since $\{g_\lambda\} \xrightarrow{\text{unif}} f$ on B', given $\zeta > 0$, there exists $N > 0$, depending upon ζ alone, such that $\lambda > N$

$$\Longrightarrow$$

$$(7.40)\ \left|\int_{y_2}^{x} [g_\lambda - f](\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda,x}(\xi, y_1), \underline{u_{\lambda,y}(\xi, y_2)})d\xi\right| < \zeta \quad ,$$

$$\left|\int_{y_2}^{x} [f - g_\lambda](\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda,x}(\xi, y_1), \underline{u_{\lambda,y}(\xi, y_1)})d\xi\right| < \zeta .$$

By hypothesis 2)',

$$(7.41)\ \left|\int_{y_2}^{x} [f(\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda,x}(\xi, y_1), \underline{u_{\lambda,y}(\xi, y_2)}) - f(\xi, y_1; u_\lambda(\xi, y_1); u_{\lambda,x}(\xi, y_1), \underline{u_{\lambda,y}(\xi, y_1)})d\xi\right|$$

$$\leqslant \int_{y_2}^{x} K\,|u_{\lambda,y}(\xi, y_2) - u_{\lambda,y}(\xi, y_1)|d\xi .$$

Moreover, since $|g_\lambda| \leqslant L$, $(\lambda = 1,2,\cdots)$,

$$(7.42)\ \left|\int_{y_1}^{y_2} g_\lambda(\xi, y_1; u_\lambda; u_{\lambda,x}, u_{\lambda,y})d\xi\right| \leqslant L\,|y_2 - y_1|$$

$$\left|\int_{y_1}^{y_2} g_\lambda(y_2, \eta; u_\lambda; u_{\lambda,x}, u_{\lambda,y})d\eta\right| \leqslant L\,|y_2 - y_1| .$$

Thus by equations (7.37) through (7.41), given $\mu > 0$, $\zeta > 0$, there exists $\delta > 0$, $N > 0$, depending only upon μ and ζ, respectively, such that $|y_2 - y_1| < \delta$ and $\lambda > N$

$\Longrightarrow$

$$(7.43)\quad \left|u_{\lambda,y}(x,y_2) - u_{\lambda,y}(x,y_1)\right| \leq K\int_{y_2}^{x} \left|u_{\lambda,y}(\xi,y_2) - u_{\lambda,y}(\xi,y_1)\right| d\xi + 4\mu + 2\zeta .$$

By Lemma 1, Chapter II, inequalities (7.35) and (7.43) imply that the sequence $\{u_{\lambda,y}\}$ is equicontinuous on R_2.

From this point on the proof is practically identical with that for Theorem 1a. Since the sequences $\{u_\lambda\}$, $\{u_{\lambda,x}\}$ and $\{u_{\lambda,y}\}$ are uniformly bounded and equicontinuous on R_2, we may apply ARZELA's theorem to obtain a subsequence of each, uniformly convergent on R_2. Hence, as for Theorem 1a, by successive extractions of subsequences we obtain a subsequence $\{u^*_\lambda\}$ of $\{u_\lambda\}$ converging uniformly on R_2 to a solution u of the integral equation

$$u(x,y) = \int_y^x d\xi \int_0^y f(\xi,\eta;\ u;\ u_x,u_y)\,d\eta ,$$

and such that for $(x,y) \in R_2$ $(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)) \in B$. The proof for Theorem 10a is now complete.

Following E. PICARD [7] p. 135 and p. 139, we show that the general statement of the mixed boundary conditions, (i.e. where u is prescribed along two intersecting curves, one characteristic and the other nowhere characteristic), can be reduced to the statement found in Theorems 10 and 10a, (i.e. where $u(x,0) = u(x,x) = 0$ for $x \in [0,\ell]$).

First, let us suppose that we prescribe

$$(7.44)\quad \begin{cases} u(x,0) = \varphi(x) \\ u(x,x) = \psi(x) \end{cases}$$

for $x \in [0,\ell]$, $\varphi(x)$ and $\psi(x) \in C'[0,\ell])$ and $\varphi(0) = \psi(0)$.

Consider

$$(7.45)\quad w(x,y) = \varphi(x) + \psi(y) - \varphi(y).$$

We have $w_{xy} = 0$ on R while

$$(7.46)\quad \begin{cases} w(x,0) = \varphi(x) \\ w(x,x) = \psi(x) \end{cases}$$

for $x \in [0,\ell]$. Hence, instead of the problem with non-homogeneous boundary conditions (7.44), by setting

$$(7.47)\quad v = u - w$$

we may consider the problem

$$(7.48)\quad \begin{cases} v_{xy} = f(x,y;\ v + w;\ v_x + w_x,\ v_y + w_y) \\ v\ (x,0) = 0 \\ v(x,x) = 0, \end{cases}$$

a problem of the type covered by Theorems 10 and 10a.

Second, suppose we prescribe u along the characteristic $y = 0$ and the nowhere characteristic curve $y = F(x)$, where $F(x) \in C'([0,\ell_1])$, $F'(x) \neq 0$ for $x \in [0,\ell_1]$ and $F(0) = 0$.

The coordinate transformation

$$(7.49)\quad \begin{cases} \bar{x} = F(x) \\ \bar{y} = y \end{cases}$$

reduces the curve $y = F(x)$ to the diagonal $\bar{y} = \bar{x}$ since the inverse F^{-1} exists and is of class C' on $[0, F(\ell_1)]$. Moreover,

$$(7.50)\quad u_{xy} = F'(x)\, u_{\bar{x}\bar{y}}.$$

Since $F'(x) \neq 0$, the form of the differential equation remains unchanged and we reduce the problem to one with initial conditions in the form (7.44).

Thus the general statement of the mixed boundary value problem for

$$(7.3) \qquad u_{xy} = f(x,y;\ u;\ u_x,u_y)$$

can be reduced to the form treated in Theorems 10 and 10a. We note that whatever continuity and Lipscnitz conditions are satisfied by (7.3) before transformation (7.49) and substitution (7.47) are satisfied as well after these operations are performed.

CHAPTER VIII

EXISTENCE THEOREMS BASED ON THE CONCEPT OF UPPER AND LOWER BOUNDING FUNCTIONS

For the ordinary differential equation $y' = f(x,y)$ with $y(x_0) = y_0$, O. PERRON [18], assuming f merely continuous, gives an existence proof that is entirely independent of the classical proofs and contains them as special cases. He bases his proof on the concept of under and over functions, defining $\varphi(x)$ to be an under function if $\varphi(x_0) = y_0$ and

(8.1) $$D_{\pm}\, \varphi(x) < f(x, \varphi(x))$$

and defining $\psi(x)$ to be an over function if $\psi(x_0) = y_0$ and

(8.2) $$D_{\pm}\, \psi(x) > f(x, \psi(x)).$$

The solutions are found to lie between the upper limit function g of the set of underfunctions and the lower limit function G of the set of overfunctions, g and G themselves being solutions.

W. MÜLLER [4] shows that PERRON's proof will not carry over directly to apply to a system.

(8.3) $$y_i = f_i(x, y_1, \cdots, y_n) \quad , \quad (i = 1, \cdots, n).$$

However, he is able to extend the classical theorem, obtaining a statement which is similar to that of PERRON and which reduces to the direct analogue of PERRON's theorem in the particular case where the functions f_i are monotonically increasing in the arguments $y_1, \cdots, y_n$.

In this chapter we return to the characteristic initial value problem for

$$u_{xy} = f(x,y;\ u;\ u_x,\ u_y). \tag{8.4}$$

We obtain results similar to those of MÜLLER above. In the following Theorems 11 and 11a we improve the statements of Theorems 1 and 1a, Chapter II, by the introduction of upper and lower bounding functions Ω and ω.

Theorem 11 (11a)

1) $f(x,y;\ u;\ p,q) \in C(T)$, $T: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \\ \omega(x,y) \leq u \leq \Omega(x,y) \\ \omega_x(x,y) \leq p \leq \Omega_x(x,y) \\ \omega_y(x,y) \leq q \leq \Omega_y(x,y) \end{cases}$

2) (2)') f is Lipschitzian (partially Lipschitzian) on T (as defined in Theorems 1 and 1a).

3) The functions $\omega(x,y)$ and $\Omega(x,y) \in C'(R)$, $R: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \end{cases}$ with $\omega_{xy}(x,y)$ and $\Omega_{xy}(x,y) \in C(R)$. Moreover,

$$\omega(x,0) = \Omega(x,0) = 0 \quad \text{for } x \in [0,\ell],$$

$$\omega(0,y) = \Omega(0,y) = 0 \quad \text{for } y \in [0,\ell],$$

and, for each $(x,y) \in R$,

$$\omega_{xy}(x,y) \leq \min_{S(x,y)} \left[f(x,y;\ u;\ p,q) \right], \tag{8.5}$$

$$\Omega_{xy}(x,y) \geq \max_{S(x,y)} \left[f(x,y;\ u;\ p,q) \right] \tag{8.6}$$

where

$$(8.7)\qquad S(x,y):\begin{cases} x = x \\ y = y \\ \omega(x,y) \leq u \leq \Omega(x,y) \\ \omega_x(x,y) \leq p \leq \Omega_x(x,y) \\ \omega_y(x,y) \leq q \leq \Omega_y(x,y) \end{cases}$$

$\Longrightarrow$ 4) (4)') There exists one and only one (at least one) function $u(x,y) \in C'(R)$, $u_{xy} \in C(R)$ such that for each $(x,y) \in R$ the point $(x,y;\ u(x,y);\ u_x(x,y)\ u_y(x,y)) \in T$, and

$$u_{xy}(x,y) = f(x,y;\ u(x,y);\ u_x(x,y),\ u_y(x,y)),$$
$$u(x,0) = u(0,y) = 0 \quad \text{for each } (x,y) \in R.$$

Proof

We extend the domain of definition of the function f over T to B': $\begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \\ -\infty < u < \infty \\ -\infty < p < \infty \\ -\infty < q < \infty \end{cases}$ by defining $f(x,y;\ u;\ p,q)$

$= f(x,y;\ \overline{u};\ \overline{p},\overline{q})$, where

$$\begin{aligned} &\overline{u} = u \text{ if } \omega(x,y) \leq u \leq \Omega(x,y), && \overline{p} = p \text{ if } \omega_x(x,y) \leq p \leq \Omega_x(x,y), \\ (8.8)\quad &\overline{u} = \omega(x,y) \text{ if } u < \omega(x,y) && \overline{p} = \omega_x(x,y) \text{ if } p < \omega_x(x,y) \\ &\overline{u} = \Omega(x,y) \text{ if } \Omega(x,y) < u && \overline{p} = \Omega_x(x,y) \text{ if } \Omega_x(x,y) < p \end{aligned}$$

and
$$\begin{aligned} \overline{q} &= q \text{ if } \omega_y(x,y) \leq q \leq \Omega_y(x,y) \\ \overline{q} &= \omega_y(x,y) \text{ if } q < \omega_y(x,y) \\ \overline{q} &= \Omega_y(x,y) \text{ if } \Omega_y(x,y) < q. \end{aligned}$$

By definition (8.8), f is uniformly continuous and uniformly bounded in B'. Moreover, by hypothesis 2)(2)') and (8.8) f satisfies a Lipschitz (partial Lipschitz) condition in B'.

Hence, by Theorem 1 (1a) Chapter II, there exists one and only one (at least one) function satisfying conclusion 4)(4)') except that for $(x,y) \in R$ we are assured only that the point $(x,y;u(x,y);u_x(x,y),u_y(x,y)) \in B'$. To complete the proof we must show that this point actually lies in T; i.e. we must show that for each $(x,y) \in R$,

$$(8.9) \quad \begin{cases} \omega(x,y) \leqslant u(x,y) \leqslant \Omega(x,y) \\ \omega_x(x,y) \leqslant u_x(x,y) \leqslant \Omega_x(x,y) \\ \omega_y(x,y) \leqslant u_y(x,y) \leqslant \Omega_y(x,y) \ . \end{cases}$$

To accomplish this, we first prove the following lemma:

Lemma 3 i) $\omega_{xy}(x,y) \leqslant u_{xy}(x,y)$ for all $(x,y) \in R$

$$\Longrightarrow \quad \begin{array}{lll} \omega(x,y) \leqslant u(x,y) & & " \\ \omega_x(x,y) \leqslant u_x(x,y) & & " \\ \omega_y(x,y) \leqslant u_y(x,y) & & " \quad , \end{array}$$

ii) $\Omega_{xy}(x,y) \geqslant u_{xy}(x,y)$ for all $(x,y) \in R$

$$\Longrightarrow \quad \begin{array}{lll} \Omega(x,y) \geqslant u(x,y) & & " \\ \Omega_x(x,y) \geqslant u_x(x,y) & & " \\ \Omega_y(x,y) \geqslant u_y(x,y) & & " \quad . \end{array}$$

Proof: For i),

$$\omega(x,y) = \int_0^x dx \int_0^y \omega_{xy} dy \leqslant \int_0^x dx \int_0^y u_{xy} dy = u(x,y)$$

$$\omega_x(x,y) = \int_0^y \omega_{xy} dy \leqslant \int_0^y u_{xy} dy = u_x(x,y)$$

$$\omega_y(x,y) = \int_0^x \omega_{xy} dx \leqslant \int_0^x u_{xy} dx = u_y(x,y).$$

The proof for ii) is analogous.

To prove (3.9) it only remains to verify that hypothesis i) and ii) of lemma 3 are satisfied by u. By hypothesis 3) and definition (3.8), for each $(x,y) \in R$,

$$\begin{aligned} \omega_{xy}(x,y) &\leq \min_{S(x,y)} \left[f(x,y;\, u;\, p,q) \right] \\ &\leq f(x,y;\, u(x,y);\, u_x(x,y),\, u_y(x,y)) \\ &= u_{xy}(x,y) \end{aligned}$$

and

$$\begin{aligned} \Omega_{xy}(x,y) &\geqslant \max_{S(x,y)} \left[f(x,y;\, u;\, p,q) \right] \\ &\geqslant f(x,y;\, u(x,y);\, u_x(x,y),\, u_y(x,y)) \\ &= u_{xy}(x,y). \end{aligned}$$

Thus, by Lemma 3, requirement (3.9) is satisfied for each $(x,y) \in R$ and the proof of Theorems 11 and 11a is complete.

It is evident upon inspection of Theorems 11 and 11a that if, instead of homogeneous initial conditions, we prescribe

$$u(x,0) = U(x) \quad \text{with } U(x) \in C'([0,\ell]),$$
$$u(0,y) = V(y) \quad \text{with } V(y) \in C'([0,\ell]),$$

where $U(0) = V(0)$, then we must require

$$\omega(x,0) = \Omega(x,0) = U(x),$$
$$\omega(0,y) = \Omega(0,y) = V(y).$$

The proof then goes through as before.

The following example is an illustration of Theorem 11:

Example 4

For the problem

$$(8.10) \quad u_{xy} = (2^{1/m} - u_x)^{1/m+1}, \; u(x,0) = u(0,y) = 0,$$

we may readily verify that

$$(8.11) \quad \omega(x,y) = \left(\frac{1}{m+1}\right)^{1/m+1} \cdot 2^{1/m(m+1)} \, xy$$

and

$$(8.12) \quad \Omega(x,y) = 2^{1/m(m+1)} \, xy$$

satisfy the hypotheses of Theorem 11 for all $x \quad 0$ and

$$0 \leq y \leq C^*_m = \frac{m}{m+1} \quad 2^{1/m+1}$$

In Chapter II we obtained the exact solution

$$(2.42) \quad u(x,y) = x \left\{ 2^{1/m} - \left[\frac{m}{m+1} (C_m - y) \right]^{m+1/m} \right\}$$

where

$$(2.43) \quad C_m = \frac{m+1}{m} \quad 2^{1/m+1}$$

is a branch point of the solution. We observe that as m increases indefinitely ω and Ω approach u from below and above, respectively, while C^*_m approaches C_m from below.

We see from this example that it is possible to obtain approximate solutions, with known limits of error, and to locate singularities in the actual solution by use of Theorem 11, provided that suitable functions ω and Ω can be obtained. For problems where explicit solutions cannot be obtained in "closed form", the procedure is to alter the right-hand side of the equation

$$u_{xy} = f(x,y;\, u;\, u_x,\, u_y)$$

so that an explicit solution of the altered equation can be ob-

tained satisfying the boundary conditions. This may lead to functions ω and Ω satisfying the hypotheses of Theorem 11. (See W. M. WHYBURN [19] and [20].) The motivation for equations (8.11) and (8.12) of Example 4 is now evident.

When we consider the possibility of applying, as explained below, the PERRON method using under and over functions to the characteristic initial value problem under consideration, we find the situation much the same as that in the case of a system of first order ordinary differential equations. We arrive at the unsatisfactory state of affairs wherein there is no assurance that the under functions remain below the over functions throughout the entire region on which a solution is known to exist. In fact, we shall presently give an example where an under function exceeds an over function within the domain of existence of a solution.

Recalling inequalities (8.1) and (8.2), we may express the application of the PERRON method as follows: We require both the under and over functions to satisfy the given characteristic initial conditions and to be continuously differentiable and to possess a mixed second derivative at each point of the domain $R: \begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \ell \end{cases}$. We further stipulate that each under function, , shall satisfy

$$(8.13) \quad \varphi_{xy}(x,y) < f(x,y;\ \varphi(x,y);\ \varphi_x(x,y),\ \varphi_y(x,y)),$$

and that each over function, ψ, shall satisfy

$$(8.14) \quad \psi_{xy}(x,y) > f(x,y;\ \psi(x,y);\ \psi_x(x,y),\ \psi_y(x,y))$$

for each $(x,y) \in R$.

Analogous arguments to those used by PERRON for the ordinary differential equation $y' = f(x,y)$ lead to the inequalities

$$\varphi_x(0,y) < \psi_x(0,y) \qquad \text{for} \qquad 0 < y \leq \ell ,$$

$$\varphi_y(x,0) < \psi_y(x,0) \qquad \text{for} \qquad 0 < x \leq \ell ,$$

for any under function φ and any over function ψ. These inequalities, together with the requirement that φ and ψ satisfy the characteristic initial data on the positive x and y axes, insure that $\psi > \varphi$ in a sufficiently small "L" shaped strip in the first quadrant adjacent to the initial characteristics. Unfortunately, this is inadequate as the following example demonstrates.

Example 5

Consider the problem

$$(8.15) \qquad u_{xy} = 0, \quad u(x,0) = u(0,y) = 0.$$

This problem has the unique solution $u \equiv 0$ throughout the finite plane. Let

$$(8.16) \begin{cases} \psi_{xy} = Ax - By^2 + C \\ \varphi_{xy} = -D , \end{cases}$$

where A, B, C and D are positive constants. By integration in (8.16) we may obtain functions ψ and φ satisfying the initial conditions of (8.15). Obviously, φ is an under function for all (x,y). Moreover, $\psi_{xy} > 0$ for all (x,y) lying in the portion of the first quadrant below the parabolic arc

$$y = +\sqrt{\frac{A}{B}\, x + \frac{C}{B}} \; ;$$

and hence ψ meets the requirements for an over function on a domain R_ℓ: $\begin{cases} 0 \leq x \leq \ell \\ 0 \leq y \leq \sqrt{\frac{C}{B}} \end{cases}$ where ℓ is arbitrarily large but finite.

Defining $h = \psi - \varphi$ we have

$$h_{xy}(x,y) = Ax - By^2 + C + D.$$

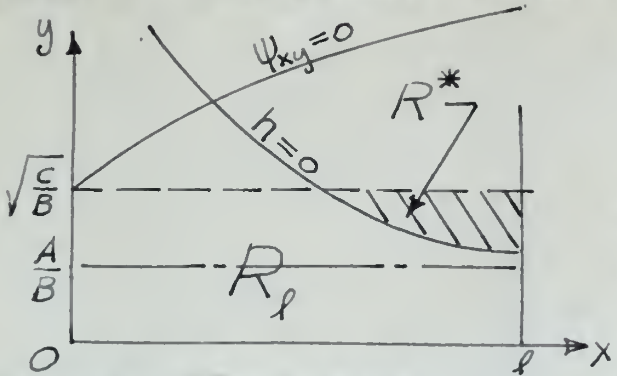
Since $h(x,0) = h(0,y) = 0$, we obtain by integration

$$h(x,y) = \frac{A}{2} x^2 y - \frac{B}{2} x^2 y^2 + (C+D)\, xy \,.$$

We note that $h > 0$ in that portion of the first quadrant below the hyperbola branch

$$y = \frac{A}{B} + \frac{2(C+D)}{Bx}$$

while $h < 0$ above this branch. From the diagram it is evident that if we require



$$\frac{A}{B} < \sqrt{\frac{C}{B}}$$

then there exists a positive constant ℓ such that within the corresponding domain R_ℓ we have a subregion R^* on which $\varphi > \psi$. Hence the PERRON method is not directly applicable to this class of problems.

Returning to Theorems 11 and 11a, we observe that if, for fixed (x,y), f is a monotonically increasing function for the arguments u, p and q, then

$$f(x,y;\, \omega(x,y);\, \omega_x(x,y),\, \omega_y(x,y)) = \min_{S(x,y)} \left[f(x,y;\, u;\, p,q) \right],$$

and

$$f(x,y;\, \Omega(x,y);\, \Omega_x(x,y),\, \Omega_y(x,y)) = \max_{S(x,y)} \left[f(x,y;u;p,q) \right].$$

In this case we may alter hypothesis 3) to require merely that

$$\omega_{xy}(x,y) \leqslant f(x,y;\ \omega(x,y);\ \omega_x(x,y),\ \omega_y(x,y))$$
$$\Omega_{xy}(x,y) \geqslant f(x,y;\ \Omega(x,y);\ \Omega_x(x,y),\ \Omega_y(x,y))$$

for each $(x,y) \in R$. This is the direct analogue to PERRON's theorem (see [18]) and corresponds to the previously mentioned result of MÜLLER for a system (8.3).

We close this chapter with the remark that Theorems 11 and 11a can be extended immediately in two ways. First, the method is directly applicable to the Cauchy problem. We require the functions ω and Ω to satisfy the Cauchy initial data and observe that the proof of Lemma 3 is essentially unchanged. Second, the method extends to apply to a system

$$u_{i,xy} = f_i(x,y;\ u_j;\ u_{j,x},\ u_{j,y}), \qquad (i = 1,\cdots,n)$$

for both characteristic and Cauchy initial value prescriptions. The modifications in the hypotheses and proof for Theorems 11 and 11a are obvious.

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